

Nash equilibrium strategy for fuzzy non-cooperative games[☆]

Li Cunlin^{a,b,*} Zhang Qiang^b

^a School of Information and Computation Sciences, Beifang University of Nationalities, Yinchuan 750021, China

^b School of Management and Economics, Beijing Institute of Technology, Beijing 100081, China

Received 9 January 2009; received in revised form 23 March 2011; accepted 28 March 2011

Available online 2 April 2011

Abstract

In this article, two-person zero-sum games are investigated in the fuzzy environment. Several models constructed by Maeda in the symmetrical fuzzy environment are extended to the models in the asymmetric fuzzy environment. The existence of equilibrium strategies for these extended models is proposed in the asymmetric fuzzy environment. However, in some cases, Nash equilibrium strategies may not exist. Therefore, two special cases are presented for which Nash equilibrium strategies do exist. In order to investigate the existence of (weak) Pareto Nash equilibrium strategies for fuzzy matrix games, we introduce the concept of crisp bi-matrix games with parameters. By solving the parametric bi-matrix games, we obtain the (weak) Pareto Nash equilibrium strategies for the fuzzy matrix games.

© 2011 Elsevier B.V. All rights reserved.

Keywords: (Fuzzy) Two-person zero-sum games; Parametric bi-matrix games; (Weak) Pareto Nash equilibrium strategy

1. Introduction

Nash presented non-cooperative game theory [1] for conflict resolution based on the assumption that each player had a well-defined quantitative utility function. In this paper, this assumption is violated because of the complexity of problems in economics, engineering, social and political sciences and the difficulty inherent in defining a well-defined quantitative payoff function for each player in these types of problems. Therefore, the gain or payoff of the players should not be modeled as a crisp number. It should rather be formulated in terms of uncertain variables, such as semantic variables, random variables, rough variables or fuzzy numbers, etc. This article uses fuzzy sets to measure the uncertainty. In other words, if the player's payoff is about 100 thousand dollars, it will be modeled as a fuzzy number.

However, the risk preference of the decision makers or experts play an important role in modeling fuzzy information. According to the prospect theory [2] presented by Kahneman and Tversky in 1979, people underweight outcomes that are merely probable in comparison with outcomes obtained with certainty. This tendency contributes to risk aversion

[☆] This work was supported by the National Natural Science Foundation of China (Nos. 71071018 and 70771010) and the Science Foundation of Beifang University of Nationalities (No. 72010Y041).

* Corresponding author at: School of Information and Computation Sciences, Beifang University of Nationalities, Yinchuan 750021, China.

E-mail address: bitlcl@gmail.com (L. Cunlin).

in choices involving sure gains and to risk seeking in choices involving sure losses. If the center of the triangular fuzzy numbers are modeled as certain outcomes, the right extension (the left extension) should be gains (losses) compared with the center. The gain does not always equal the loss because the decision makers or experts have different risk preferences. Therefore, this article models the fuzzy information as asymmetric fuzzy numbers; in other words, the expected payoffs of the players are assumed to be fuzzy numbers.

Many excellent works have contributed to the theory of non-cooperative fuzzy games, beginning with the work of Butnariu [3]. He assigned a strategic composition for any crisp strategies and determined the equilibrium strategies based on the fuzzy preference relations of the investment of the player's pure strategy. Campos [4] modeled two-person zero-sum games with fuzzy payoffs based on the fuzzy order and transformed the fuzzy games into fuzzy optimization problems. Using Yager's fuzzy numbers ordering method, he transformed the problem of finding a solution to the fuzzy matrix games into a linear programming problem [5]. Sakawa and Nishizaki [6,7] investigated single-objective and multi-objective games with fuzzy goals and fuzzy payoffs. They introduced a fuzzy goal for a payoff and assumed that every player tried to maximize his degree of attainment of the fuzzy goal. Then, the equilibrium solutions with respect to the degree of attainment of a fuzzy goal were defined. Finally, they transformed their models into a fractional programming problem and solved the fractional programming problem by a relaxed method. Vijay, Bector and Chandra [8–11] proposed non-cooperative games in uncertainty based on the duality in fuzzy linear programming. They defined two types of solutions for the game and solved them by transforming the fuzzy dual problems into pairs of crisp dual problems. Larbani [12–14] proposed a new class of games with fuzzy parameters involving two aspects: the game aspect and a decision making under uncertainty aspect. When the players choose their strategies, they must consider both the behavior of the other players and the possible realizations of the fuzzy parameter. Maeda [15,16] presented three kinds of equilibrium strategies for fuzzy matrix games based on specific symmetric triangular fuzzy numbers and investigated the existence conditions of equilibrium strategies for these models. However, it was hard to accept that fuzzy payoffs were modeled as symmetric triangular fuzzy numbers.

This paper will generalize Maeda's model, which was established based on symmetrical triangular fuzzy numbers, and it will investigate all types of equilibrium strategies based on more general asymmetrical triangular fuzzy numbers. In order to find the (weak) Pareto Nash equilibrium strategies of games with fuzzy payoffs, the crisp bi-matrix games with parameters and their equilibrium strategies are given. For that purpose, this paper is organized as follows. In Section 2, we introduce some basic definitions and notations on fuzzy set theory, followed by several different fuzzy ordering methods (including the fuzzy maximum order presented by Ramík presented [17]). In Section 3, the fuzzy zero-sum non-cooperative games and some useful notations about game theory are presented. Then, we focus on finding the existence conditions for different equilibrium strategies and establishing the relation between the matrix games with fuzzy payoffs and the crisp matrix games with parameters. In the last section, several examples are given to illustrate our conclusions.

2. Preliminaries

In this section, we summarize some basic concepts of fuzzy sets, which were initiated by Zadeh [18,19] to measure fuzzy events.

Definition 2.1. A fuzzy number \tilde{a} is defined as a fuzzy set on the space of real numbers \mathcal{R} , whose membership function $\mu_{\tilde{a}}(x)$ satisfies the following conditions:

- (i) $\mu_{\tilde{a}}(x)$ is a mapping from \mathcal{R} to the closed interval $[0,1]$;
- (ii) there exists a unique real number c , called the center of \tilde{a} , such that
 - (a) $\mu_{\tilde{a}}(c) = 1$,
 - (b) $\mu_{\tilde{a}}(x)$ is nondecreasing on $(-\infty, c]$,
 - (c) $\mu_{\tilde{a}}(x)$ is nonincreasing on $[c, \infty)$.

In the rest of this paper, the α -level sets of \tilde{a} are defined for $\tilde{a} \in \mathcal{F}$ and $\alpha \in [0, 1]$ as $\tilde{a}_\alpha \triangleq \{x | \mu_{\tilde{a}}(x) \geq \alpha, x \in \mathcal{R}\}$. $\tilde{a}_0 \triangleq \{x | \mu_{\tilde{a}}(x) > 0, x \in \mathcal{R}\}$ is called the support of \tilde{a} . $a_\alpha^R \triangleq \sup \tilde{a}_\alpha$, $a_\alpha^L \triangleq \inf \tilde{a}_\alpha$, and $\tilde{a}_\alpha = [a_\alpha^L, a_\alpha^R]$.

Definition 2.2. Let \tilde{a} be a fuzzy number. If the membership function of \tilde{a} is given by

$$\mu_{\tilde{a}}(x) = \begin{cases} \frac{x - a + l}{l}, & x \in [a - l, a], \\ \frac{a + r - x}{r}, & x \in [a, a + r], \\ 0 & \text{otherwise,} \end{cases}$$

where, a , l and r are all real numbers, and l and r are non-negative. Then, \tilde{a} is called a triangular fuzzy number, denoted by $\tilde{a} = (a, l, r)$. We denote the sets of triangular fuzzy numbers as \mathcal{F} .

Definition 2.3. Let $\tilde{a}, \tilde{b} \in \mathcal{F}$, $c \in \mathcal{R}$; Then, the membership function of the sum of \tilde{a} and \tilde{b} and the scalar product of c and \tilde{a} are defined as follows:

- (i) $\mu_{\tilde{a}+\tilde{b}}(x) = \sup_{x=\mu+\nu} \min\{\mu_{\tilde{a}}, \mu_{\tilde{b}}\}$,
- (ii) $\mu_{c\tilde{a}}(x) = \max\{\sup_{x=c\mu}, 0\}$, with $\sup\{\phi\} = -\infty$.

Lemma 2.1. Let $\tilde{a} = (a, l, r)$, $\tilde{b} = (b, m, n) \in \mathcal{F}$, $c \in \mathcal{R}^+$. It holds that

- (i) $c\tilde{a} = (ca, cl, cr)$,
- (ii) $\tilde{a} + \tilde{b} = (a + b, l + m, r + n)$.

Definition 2.4. Let $x = (x_1, x_2, \dots, x_n)$, $y = (y_1, y_2, \dots, y_n)$ be vectors in n -dimensional Euclidean space \mathcal{R}^n .

- (i) $x \leq y$ if and only if $x_i \leq y_i$ holds with $x_i, y_i \in \mathcal{R}$, $i = 1, 2, \dots, n$,
- (ii) $x \leq y$ if and only if $x \leq y$ and $x \neq y$ hold,
- (iii) $x < y$ if and only if $x_i < y_i$ holds with $x_i, y_i \in \mathcal{R}$, $i = 1, 2, \dots, n$.

Definition 2.5 (Maeda [15]). Let $\tilde{a}, \tilde{b} \in \mathcal{F}$ be fuzzy numbers. Then,

- (i) \tilde{b} dominates \tilde{a} (denoted by $\tilde{a} \lesssim \tilde{b}$) if and only if $(a_{\alpha}^L, a_{\alpha}^R) \leq (b_{\alpha}^L, b_{\alpha}^R)$ hold with $\alpha \in [0, 1]$;
- (ii) \tilde{b} strictly dominates \tilde{a} (denoted by $\tilde{a} \prec \tilde{b}$) if and only if $(a_{\alpha}^L, a_{\alpha}^R) \leq (b_{\alpha}^L, b_{\alpha}^R)$ hold with $\alpha \in [0, 1]$;
- (iii) \tilde{b} strongly dominates \tilde{a} (denoted by $\tilde{a} < \tilde{b}$) if and only if $(a_{\alpha}^L, a_{\alpha}^R) < (b_{\alpha}^L, b_{\alpha}^R)$ hold with $\alpha \in [0, 1]$;
- (iv) \tilde{b} is equal to \tilde{a} (denoted by $\tilde{a} = \tilde{b}$) if and only if $(a_{\alpha}^L, a_{\alpha}^R) = (b_{\alpha}^L, b_{\alpha}^R)$ hold with $\alpha \in [0, 1]$.

It is obvious that (i) of Definition 2.5 is precisely equivalent to the fuzzy maximum order given in [17].

Theorem 2.1. Let $\tilde{a} = (a, l, r)$, $\tilde{b} = (b, m, n) \in \mathcal{F}$, where a, b, l, m, r and n are all real numbers, and l, m, r and n are all non-negatives; then,

- (i) $\tilde{a} \lesssim \tilde{b}$ if and only if $\max\{m - l, 0\} \leq b - a$ and $\max\{r - n, 0\} \leq b - a$;
- (ii) $\tilde{a} < \tilde{b}$ if and only if $\max\{m - l, 0\} < b - a$ and $\max\{r - n, 0\} < b - a$ hold.

The proof is omitted.

3. Several Nash equilibrium strategies for a two-person zero-sum game with fuzzy payoffs

Next, we introduce some useful definitions for non-cooperative games. Then, we focus on the games with fuzzy payoffs.

Definition 3.1. Let $M = \{1, 2, \dots, m\}$ and $N = \{1, 2, \dots, n\}$ be the sets of pure strategies of Player I and Player J , respectively. The mixed strategies of Player I and Player J are probability distributions on the sets of pure strategies,

represented by

$$S_I = \left\{ (x_1, x_2, \dots, x_m) \in \mathcal{R}^m \mid x_i \geq 0, i = 1, 2, \dots, m, \sum_{i=1}^m x_i = 1 \right\},$$

$$S_J = \left\{ (x_1, x_2, \dots, x_n) \in \mathcal{R}^n \mid x_i \geq 0, i = 1, 2, \dots, n, \sum_{i=1}^n x_i = 1 \right\},$$

respectively, where \mathcal{R}^m is the m -dimensional real vector space.

In this paper, the payoffs of the pair $(i, j) \in S_I \times S_J$ are modeled as asymmetric triangular fuzzy numbers $\tilde{a} = (a, l, r) \in \mathcal{F}$. The membership functions of the fuzzy payoffs are given based on the experts assessments.

Definition 3.2. Let $\tilde{a}_{ij} \in \mathcal{F}$ represent the incomes of Player I and the losses of Player J based on the pair $(i, j) \in S_I \times S_J$. We then say that the game is a fuzzy two-person zero-sum game. The fuzzy payoff matrix of the game is given by

$$\tilde{A} = \begin{pmatrix} \tilde{a}_{11} & \cdots & \tilde{a}_{1n} \\ \vdots & \ddots & \vdots \\ \tilde{a}_{m1} & \cdots & \tilde{a}_{mn} \end{pmatrix},$$

where $\tilde{a}_{ij} = (a_{ij}, l_{ij}, r_{ij}) \in \mathcal{F}$. The fuzzy two-person zero-sum games are denoted by $\tilde{\Gamma} \equiv (\{I, J\}, S_I, S_J, \tilde{A})$ and are also called fuzzy matrix games.

The expected payoffs of the players are $E(x, y) = x^T \tilde{A} y = \sum_{i=1}^m \sum_{j=1}^n x_i \tilde{a}_{ij} y_j$. By Lemma 1 of Ref. [17], $E(x, y)$ is still a triangular fuzzy number. The fuzzy matrix represents the fuzzy expected income matrix of Player I and the fuzzy expected loss matrix of Player J . In the rest of this paper, we denote $A = (a_{ij})_{m \times n}$, $L = (l_{ij})_{m \times n}$, $R = (r_{ij})_{m \times n}$, $A_0^L = A - L$ and $A_0^R = A + R$.

Definition 3.3 (Maeda [15]). A pair $(x^*, y^*) \in S_I \times S_J$ is a Nash equilibrium strategy for a game $\tilde{\Gamma}$ if

- (i) $x^T \tilde{A} y^* \preceq x^{*T} \tilde{A} y^*$, $x \in S_I$;
- (ii) $x^{*T} \tilde{A} y^* \preceq x^{*T} \tilde{A} y$, $y \in S_J$.

Definition 3.4 (Maeda [15]). A pair $(x^*, y^*) \in S_I \times S_J$ is a Pareto Nash equilibrium strategy for a game $\tilde{\Gamma}$, if

- (i) there exists no $x \in S_I$ such that $x^{*T} \tilde{A} y^* \prec x^T \tilde{A} y^*$;
- (ii) there exists no $y \in S_J$ such that $x^{*T} \tilde{A} y \prec x^{*T} \tilde{A} y^*$.

Definition 3.5 (Maeda [15]). A pair $(x^*, y^*) \in S_I \times S_J$ is a weak Pareto Nash equilibrium strategy for a game $\tilde{\Gamma}$ if

- (i) there exists no $x \in S_I$ such that $x^{*T} \tilde{A} y^* < x^T \tilde{A} y^*$;
- (ii) there exists no $y \in S_J$ such that $x^{*T} \tilde{A} y < x^{*T} \tilde{A} y^*$.

Clearly, if the elements of \tilde{A} are all crisp number, the above three definitions then coincide with the Nash equilibrium strategy for crisp two-person zero-sum games. Therefore, these definitions are natural generalizations of crisp two-person zero-sum games.

Theorem 3.1. A pair $(x^*, y^*) \in S_I \times S_J$ is the Nash equilibrium strategy for a fuzzy two-person zero-sum game $\tilde{\Gamma} \equiv (\{I, J\}, S_I, S_J, \tilde{A})$ if and only if the following inequalities hold with $x \in S_I, y \in S_J$:

$$x^T A y^* \leq x^{*T} A y^* \leq x^{*T} A y, \tag{1}$$

$$x^T A y^* - x^T L y^* \leq x^{*T} A y^* - x^{*T} L y^* \leq x^{*T} A y - x^{*T} L y, \tag{2}$$

$$x^T A y^* + x^T R y^* \leq x^{*T} A y^* + x^{*T} R y^* \leq x^{*T} A y + x^{*T} R y. \tag{3}$$

Proof. Let the pair $(x^*, y^*) \in S_I \times S_J$ be a Nash equilibrium strategy for the game $\tilde{\Gamma}$. According to Theorem 2.1 and Definition 3.3(i), we have

$$\max\{x^{*T}Ly^* - x^TLy^*, 0\} \leq x^{*T}Ay^* - x^TAy^*, \quad (4)$$

$$\max\{x^TRy^* - x^{*T}Ry^*, 0\} \leq x^{*T}Ay^* - x^TAy^*. \quad (5)$$

From (4), it follows that

$$x^{*T}Ly^* - x^TLy^* \leq x^{*T}Ay^* - x^TAy^*, \quad 0 \leq x^{*T}Ay^* - x^TAy^*.$$

By rearranging, we get

$$x^TAy^* - x^TLy^* \leq x^{*T}Ay^* - x^{*T}Ly^*, \quad x^TAy^* \leq x^{*T}Ay^*. \quad (6)$$

Analogously, (5) implies that

$$x^TRy^* - x^{*T}Ry^* \leq x^{*T}Ay^* - x^TAy^*.$$

By rearranging, we get

$$x^TRy^* + x^TAy^* \leq x^{*T}Ay^* + x^{*T}Ry^*. \quad (7)$$

Analogously, from Theorem 2.1 and Definition 3.3(ii), we have

$$x^{*T}Ay^* - x^{*T}Ly^* \leq x^{*T}Ay - x^{*T}Ly, \quad x^{*T}Ay^* \leq x^{*T}Ay. \quad (8)$$

$$x^{*T}Ay^* + x^{*T}Ry^* \leq x^{*T}Ry + x^{*T}Ay. \quad (9)$$

By rearranging (6)–(9), we get (1)–(3).

Otherwise, according to (1),

$$0 \leq x^{*T}Ay^* - x^TAy^*. \quad (10)$$

From (2), it follows that

$$x^TAy^* - x^TLy^* \leq x^{*T}Ay^* - x^{*T}Ly^*, \quad (11)$$

$$x^{*T}Ay^* - x^{*T}Ly^* \leq x^{*T}Ay - x^{*T}Ly. \quad (12)$$

From (3), it follows that

$$x^TAy^* + x^TRy^* \leq x^{*T}Ay^* + x^{*T}Ry^*, \quad (13)$$

$$x^{*T}Ay^* + x^{*T}Ry^* \leq x^{*T}Ay + x^{*T}Ry. \quad (14)$$

Combining (10) with (11), we obtain (4). Combining (10) with (13), we get (5). Therefore, we have Definition 3.3(i) by Theorem 2.1.

Analogously, from Theorem 2.1 and the results of rearranging (10), (13) and (14), we obtain Definition 3.3(ii). \square

By the above result, combining $x^TAy - x^TLy = x^TA_0^L y$ and $x^TAy + x^TRy = x^TA_0^R y$, we conclude that fuzzy two-person zero-sum games are equivalent to the following three matrix games $\Gamma_l \equiv (\{I, J\}, S_I, S_J, A_0^L)$, $\Gamma_c \equiv (\{I, J\}, S_I, S_J, A)$, $\Gamma_r \equiv (\{I, J\}, S_I, S_J, A_0^R)$. If we denote these three two-person zero-sum matrix games Γ_l , Γ_c , and Γ_r as a triple (A_0^L, A, A_0^R) , then the following corollary holds.

Corollary 3.1. *The pair $(x^*, y^*) \in S_I \times S_J$ is the Nash equilibrium strategy for the game $\tilde{\Gamma}$ if and only if the pair is the Nash equilibrium strategy of the triple (A_0^L, A, A_0^R) , that is,*

$$x^T(A_0^L, A, A_0^R)y^* \leq x^{*T}(A_0^L, A, A_0^R)y^* \leq x^{*T}(A_0^L, A, A_0^R)y$$

where $x^T(A_0^L, A, A_0^R)y = (x^TA_0^L y, x^TAy, x^TA_0^R y)$.

We have presented equivalent conditions for Nash equilibrium strategies in the fuzzy environment. Unfortunately, we have found that the games with fuzzy payoffs do not always have a Nash equilibrium strategy, such as in Example 3.1. However, in the following cases, there exists at least one Nash equilibrium strategy.

Definition 3.6. Let $\tilde{A} = (\tilde{a}_{ij})_{m \times n}$ be the fuzzy payoff matrix of game $\tilde{\Gamma}$; it is said to be a proportional fuzzy game if and only if there exists $k_1, k_2 \in (0, 1]$ such that

$$k_1 a_{ij} = l_{ij}, \quad k_2 a_{ij} = r_{ij}, \quad i = 1, 2, \dots, m, \quad j = 1, 2, \dots, n.$$

Theorem 3.2. Let $\tilde{A} = (\tilde{a}_{ij})_{m \times n}$ be the payoff matrix of a proportional fuzzy game $\tilde{\Gamma}$; the pair $(x^*, y^*) \in S_I \times S_J$ is the Nash equilibrium strategy for the game $\tilde{\Gamma}$ if and only if (x^*, y^*) is still the Nash equilibrium strategy for the game Γ_c .

The proof is omitted.

Definition 3.7. Let $\tilde{A} = (\tilde{a}_{ij})_{m \times n}$ be the fuzzy payoff matrix of game $\tilde{\Gamma}$; it is said to be a constant extension fuzzy game if and only if there exists $l, r \in \mathcal{R}$ such that

$$l_{ij} = l, \quad r_{ij} = r, \quad i = 1, 2, \dots, m, \quad j = 1, 2, \dots, n.$$

Theorem 3.3. Let $\tilde{A} = (\tilde{a}_{ij})_{m \times n}$ be the payoff matrix of a constant extension fuzzy game $\tilde{\Gamma}$; the pair $(x^*, y^*) \in S_I \times S_J$ is the Nash equilibrium strategy for the game $\tilde{\Gamma}$ if and only if (x^*, y^*) is still the Nash equilibrium strategy for the game Γ_c .

Proof. From Definition 3.7, L and R are both constant matrices. Therefore, by Lemma 1 of Ref. [17],

$$\tilde{a}_{ij} = (l, a_{ij}, r) = a_{ij} + (l, 0, r) = a_{ij} + \tilde{d},$$

where $\tilde{d} = (l, 0, r)$. Hence, for $x \in S_I, y \in S_J$, we obtain the following equations:

$$x^T \tilde{A} y = \sum_{i=1}^m \sum_{j=1}^n x_i \tilde{a}_{ij} y_j = \sum_{i=1}^m \sum_{j=1}^n x_i (a_{ij} + \tilde{d}) y_j = \sum_{i=1}^m \sum_{j=1}^n x_i a_{ij} y_j + \sum_{i=1}^m \sum_{j=1}^n x_i \tilde{d} y_j = x^T A y + \tilde{d}. \tag{15}$$

Next, let $(x^*, y^*) \in S_I \times S_J$ be the Nash equilibrium strategy of the game Γ_c ; it follows that

$$x^{*T} A y^* \leq x^{*T} A y^* \leq x^{*T} A y.$$

Combining Definition 2.5 with Lemma 1 of Ref. [17], it is obvious that the above inequalities are equivalent to

$$x^{*T} A y^* + \tilde{d} \preceq x^{*T} A y^* + \tilde{d} \preceq x^{*T} A y + \tilde{d}.$$

Using (15), this is also equivalent to

$$x^{*T} \tilde{A} y^* \preceq x^{*T} \tilde{A} y^* \preceq x^{*T} \tilde{A} y.$$

In other words, the pair (x^*, y^*) satisfies Definition 3.3.

And vice versa.

Thus, Theorem 3.3 is proved. \square

Theorem 3.4. Let $\tilde{\Gamma} \equiv (\{I, J\}, S_I, S_J, \tilde{A})$ be fuzzy two-person zero-sum games. A pair $(x^*, y^*) \in S_I \times S_J$ is the Pareto Nash equilibrium strategy for the game $\tilde{\Gamma}$ if and only if

(i) there exists no $x \in S_I$ such that

$$x^{*T} (A_0^L, A_0^R) y^* \leq x^T (A_0^L, A_0^R) y^* \quad \text{and} \quad x^{*T} A y^* \leq x^T A y^*; \tag{16}$$

(ii) there exists no $y \in S_J$ such that

$$x^{*T}(A_0^L, A_0^R)y \leq x^{*T}(A_0^L, A_0^R)y^* \quad \text{and} \quad x^{*T}Ay \leq x^{*T}Ay^*. \quad (17)$$

Proof. Let the pair $(x^*, y^*) \in S_I \times S_J$ be the Pareto Nash equilibrium for the game $\tilde{\Gamma}$. First, we assume that there exists a strategy $\bar{x} \in S_I$ such that (16) holds, that is,

$$x^{*T}(A - L, A + R)y^* \leq \bar{x}^T(A - L, A + R)y^* \quad \text{and} \quad x^{*T}Ay^* \leq \bar{x}^TAy^*. \quad (18)$$

As above, this implies the following:

$$x^{*T}(A - L)y^* \leq \bar{x}^T(A - L)y^*, \quad x^{*T}(A + R)y^* \leq \bar{x}^T(A + R)y^*.$$

Furthermore, the above two inequalities do not occur simultaneously. Thus, for $\alpha \in [0, 1)$, from the above inequalities, it follows that

$$\begin{aligned} & (x^{*T}(\alpha A + (1 - \alpha)(A - L))y^*, x^{*T}(\alpha A + (1 - \alpha)(A + R))y^*) \\ & \leq (\bar{x}^T(\alpha A + (1 - \alpha)(A - L))y^*, \bar{x}^T(\alpha A + (1 - \alpha)(A + R))y^*). \end{aligned}$$

By rearranging, this implies that

$$(x^{*T}(A - (1 - \alpha)L)y^*, x^{*T}(A + (1 - \alpha)R)y^*) \leq (\bar{x}^T(A - (1 - \alpha)L)y^*, \bar{x}^T(A + (1 - \alpha)R)y^*),$$

that is, $x^{*T}\tilde{A}y^* \leq \bar{x}^T\tilde{A}y^*$, by Definition 2.5. This is a contradiction.

Analogously, there exists no y such that (17) holds.

Otherwise, let the pair $(x^*, y^*) \in S_I \times S_J$ satisfy (16) and (17). Assume that there exists a strategy $\bar{x} \in S_I$ such that $x^{*T}\tilde{A}y^* \leq \bar{x}^T\tilde{A}y^*$. From Definition 2.5, for $\alpha \in [0, 1)$, it is clear that

$$x^{*T}(A_\alpha^L, A_\alpha^R)y^* \leq \bar{x}^T(A_\alpha^L, A_\alpha^R)y^*$$

furthermore, A_α^L and A_α^R are continuous with respect to α . Therefore, let α tend to 1, it follows

$$x^{*T}Ay^* \leq \bar{x}^TAy^*.$$

Set $\alpha = 0$, it follows

$$x^{*T}(A_0^L, A_0^R)y^* \leq \bar{x}^T(A_0^L, A_0^R)y^*.$$

These contradict (16).

Analogously, it is clear that there exists no $y \in S_J$ such that $x^{*T}\tilde{A}y \leq x^{*T}\tilde{A}y^*$. \square

Theorem 3.5. Let $\tilde{\Gamma} \equiv (\{I, J\}, S_I, S_J, \tilde{A})$ be fuzzy two-person zero-sum games. A pair $(x^*, y^*) \in S_I \times S_J$ is the weak Pareto Nash equilibrium strategy for the game $\tilde{\Gamma}$ if and only if

(i) there exists no $x \in S_I$ such that

$$x^{*T}(A_0^L, A, A_0^R)y^* < x^T(A_0^L, A, A_0^R)y^*;$$

(ii) there exists no $y \in S_J$ such that

$$x^{*T}(A_0^L, A, A_0^R)y < x^{*T}(A_0^L, A, A_0^R)y^*.$$

Proof. The proof of this theorem is similar to that of Theorem 3.6. \square

Next, parametric bi-matrix games will be characterized, and other types of Nash equilibrium strategies will be investigated through parametric bi-matrix games.

Let $\tilde{\Gamma} \equiv (\{I, J\}, S_I, S_J, \tilde{A})$ be fuzzy two-person zero-sum games, and suppose that Player I chooses pure strategy i and Player J chooses pure strategy j ; we set $(1 - \lambda)(a_{ij} - l_{ij}) + \lambda(a_{ij} + r_{ij})$ to be payoffs of Player I and $-[(1 - \mu)(a_{ij} - l_{ij}) + \mu(a_{ij} + r_{ij})]$ to be the payoffs of Player J , where $\lambda, \mu \in [0, 1]$. The payoff matrices of Player I and Player J are

$$A(\lambda) = (1 - \lambda)(A - L) + \lambda(A + R) \quad \text{and} \quad -A(\mu) = -[(1 - \mu)(A - L) + \mu(A + R)].$$

Then, we have the parametric bi-matrix game $\Gamma(\lambda, \mu) \equiv (\{I, J\}, S_I, S_J, A(\lambda), -A(\mu))$, where λ and μ are parameters.

Definition 3.8. A pair $(x^*, y^*) \in S_I \times S_J$ is the Nash equilibrium strategy of the parametric bi-matrix game $\Gamma(\lambda, \mu)$ with $\lambda, \mu \in [0, 1]$, if

- (i) $x^T A(\lambda)y^* \leq x^{*T} A(\lambda)y^*, x \in S_I$;
- (ii) $x^{*T} A(\mu)y^* \leq x^{*T} A(\mu)y, y \in S_J$.

Lemma 3.1. *There exists at least one Nash equilibrium strategy for all parametric bi-matrix games $\Gamma(\lambda, \mu)$ with $\lambda, \mu \in [0, 1]$.*

Theorem 3.6. *Let the pair $(x^*, y^*) \in S_I \times S_J$ be the Nash equilibrium strategy for the parametric bi-matrix game $\Gamma(\lambda, \mu)$ with $\lambda, \mu \in (0, 1)$. Then, (x^*, y^*) is the Pareto Nash equilibrium strategy for the game $\tilde{\Gamma}$.*

Proof. Let (x^*, y^*) be the Nash equilibrium strategy for the parametric bi-matrix game $\Gamma(\lambda_0, \mu_0)$, where $\lambda_0, \mu_0 \in (0, 1)$. For $x \in S_I$, from Definition 3.8(i), it follows that

$$(1 - \lambda_0)x^T A_0^L y^* + \lambda_0 x^T A_0^R y^* \leq (1 - \lambda_0)x^{*T} A_0^L y^* + \lambda_0 x^{*T} A_0^R y^*. \tag{19}$$

For $y \in S_J$, from Definition 3.8(ii), it follows that

$$(1 - \mu_0)x^{*T} A_0^L y + \mu_0 x^{*T} A_0^R y \leq (1 - \mu_0)x^{*T} A_0^L y^* + \mu_0 x^{*T} A_0^R y^*. \tag{20}$$

First, we assume there exists $\bar{x} \in S_I$ such that $x^{*T} \tilde{A}y^* \prec \bar{x}^T \tilde{A}y^*$ holds. From Definition 2.5, it follows that

$$(x^{*T} A_0^L y^*, x^{*T} A_0^R y^*) \leq (\bar{x}^T A_0^L y^*, \bar{x}^T A_0^R y^*).$$

Because $x^{*T} A_0^L y^* = \bar{x}^T A_0^L y^*$ and $x^{*T} A_0^R y^* = \bar{x}^T A_0^R y^*$ do not occur simultaneously, we have

$$(1 - \lambda_0)x^{*T} A_0^L y^* + \lambda_0 x^{*T} A_0^R y^* < (1 - \lambda_0)\bar{x}^T A_0^L y^* + \lambda_0 \bar{x}^T A_0^R y^*, \lambda \in (0, 1).$$

This contradicts (19).

Analogously, we conclude that (x^*, y^*) satisfies Definition 3.8(ii). Thus, the theorem is proved. \square

Theorem 3.7. *Let the pair $(x^*, y^*) \in S_I \times S_J$ be the Nash equilibrium strategy for the parametric bi-matrix game $\Gamma(\lambda, \mu)$ with $\lambda, \mu \in [0, 1]$. Then, (x^*, y^*) is the weak Pareto Nash equilibrium strategy for the game $\tilde{\Gamma}$.*

Proof. The proof of this theorem is similar to that of the Theorem 3.6. \square

From Theorems 3.6, 3.7 and Lemma 3.1, it is easy to conclude the following theorem.

Theorem 3.8. *Let $\tilde{\Gamma}$ be two-person zero-sum games with fuzzy payoffs. The following properties hold:*

- (i) *there exists at least one Pareto Nash equilibrium strategy for the game $\tilde{\Gamma}$;*
- (ii) *there exists at least one weak Pareto Nash equilibrium strategy for the game $\tilde{\Gamma}$.*

Example 3.1. Let \tilde{A} be the fuzzy payoff matrix of the fuzzy two-person zero-sum game $\tilde{\Gamma}$, given as follows:

$$\tilde{A} = \begin{pmatrix} (100, 10, 15) & (85, 5, 10) \\ (80, 5, 5) & (185, 15, 20) \end{pmatrix}.$$

Find the Nash equilibrium strategies, Pareto Nash equilibrium strategies and the weak Pareto Nash equilibrium strategies for the game $\tilde{\Gamma}$.

We denote by $x^T = (t, 1 - t)$ and $y^T = (s, 1 - s)$ the mixed strategies of Player I and Player J , respectively. It is easy to show that there is no $(x, y) \in S_I \times S_J$ satisfying (1), (2) and (3) at the same time. Hence, there is no Nash equilibrium strategy for the game $\tilde{\Gamma}$.

Then, for $\Gamma(\lambda, \mu)$ with $\lambda, \mu \in [0, 1]$, we define two payoff matrices as follows:

$$A(\lambda) = \begin{pmatrix} 90 + 25\lambda & 80 + 15\lambda \\ 75 + 10\lambda & 170 + 35\lambda \end{pmatrix}, \quad A(\mu) = \begin{pmatrix} 90 + 25\mu & 80 + 15\mu \\ 75 + 10\mu & 170 + 35\mu \end{pmatrix}.$$

It is easy to see that the Nash equilibrium strategies for the parametric bi-matrix game $\Gamma(\lambda, \mu)$ satisfy the following:

$$\begin{aligned} (1, 0)A(\lambda)y &\leq x^T A(\lambda)y, & (0, 1)A(\lambda)y &\leq x^T A(\lambda)y, \\ x^T A(\mu)y &\leq x^T A(\mu)(0, 1)^T, & x^T A(\mu)y &\leq x^T A(\mu)(1, 0)^T. \end{aligned}$$

The above inequalities are equivalent to

$$\begin{cases} (105 + 35\lambda)(1 - t)s - (90 + 20\lambda)(1 - t) \leq 0, \\ (105 + 35\lambda)st - (90 + 20\lambda)t \geq 0. \end{cases}$$

$$\begin{cases} (105 + 35\mu)(1 - s)t - (95 + 25\mu)(1 - s) \geq 0, \\ (105 + 35\mu)st - (95 + 25\mu)s \leq 0. \end{cases}$$

Thus, for $\lambda, \mu \in (0, 1)$, Nash equilibrium strategies for the parametric bi-matrix game $\Gamma(\lambda, \mu)$ are as follows:

$$x^{*T} = (x_1^*, x_2^*) = \left(\frac{95 + 25\mu}{105 + 35\mu}, \frac{10 + 10\mu}{105 + 35\mu} \right), \quad y^{*T} = (y_1^*, y_2^*) = \left(\frac{90 + 20\lambda}{105 + 35\lambda}, \frac{15 + 15\lambda}{105 + 35\lambda} \right).$$

By Theorems 3.6 and 3.7, the Pareto Nash equilibrium strategies and the weak Pareto Nash equilibrium strategies for the game $\tilde{\Gamma}$ are as follows:

$$PN = \left\{ (x^{*T}, y^{*T}) = \left(\left(\frac{95 + 25\mu}{105 + 35\mu}, \frac{10 + 10\mu}{105 + 35\mu} \right), \left(\frac{90 + 20\lambda}{105 + 35\lambda}, \frac{15 + 15\lambda}{105 + 35\lambda} \right) \right) \mid \lambda, \mu \in (0, 1) \right\},$$

$$WPN = \left\{ (x^{*T}, y^{*T}) = \left(\left(\frac{95 + 25\mu}{105 + 35\mu}, \frac{10 + 10\mu}{105 + 35\mu} \right), \left(\frac{90 + 20\lambda}{105 + 35\lambda}, \frac{15 + 15\lambda}{105 + 35\lambda} \right) \right) \mid \lambda, \mu \in [0, 1] \right\}.$$

Example 3.2. Let \tilde{A} be the payoff matrix of the fuzzy two-person zero-sum game $\tilde{\Gamma}$, given as follows:

$$\tilde{A} = \begin{pmatrix} (70, 7, 14) & (90, 9, 18) \\ (110, 11, 22) & (30, 3, 6) \end{pmatrix}.$$

Find the Nash equilibrium strategies for the game $\tilde{\Gamma}$.

It is obvious that $\tilde{\Gamma}$ is a proportional fuzzy game. From Theorem 3.2, it is easy to show that the Nash equilibrium strategy and the value of game $\tilde{\Gamma}$ are given by

$$\left(\left(\left(\frac{4}{5}, \frac{1}{5} \right), \left(\frac{3}{5}, \frac{2}{5} \right) \right), v_{\tilde{\Gamma}} = \left(\frac{4}{5}, \frac{1}{5} \right) \tilde{A} \left(\frac{3}{5}, \frac{2}{5} \right)^T = (74, 7.4, 14.8) \right).$$

Example 3.3. Let \tilde{A} be the payoff matrix of the fuzzy two-person zero-sum games $\tilde{\Gamma}$, given as follows:

$$\tilde{A} = \begin{pmatrix} (40, 7, 4) & (70, 7, 4) \\ (50, 7, 4) & (20, 7, 4) \end{pmatrix}.$$

Find the Nash equilibrium strategies for the game $\tilde{\Gamma}$.

It is obvious that \tilde{I} is a constant extension fuzzy game. From Theorem 3.3, it is easy to show that the Nash equilibrium strategy and the value of game \tilde{I} are given by

$$\left(\left(\frac{1}{2}, \frac{1}{2} \right), \left(\frac{5}{6}, \frac{1}{6} \right) \right), \quad v_{\tilde{I}} = \left(\frac{1}{2}, \frac{1}{2} \right) \tilde{A} \left(\frac{5}{6}, \frac{1}{6} \right)^T = \left(\frac{130}{3}, 7, 4 \right).$$

4. Conclusion

In this paper, we have developed a new theoretical approach to two-person zero-sum games with fuzzy payoffs. We then examined some extensions of Maeda's models and generalized existence conditions for all types of equilibrium strategy from models based on symmetric triangular numbers to models based on asymmetric triangular numbers. By investigating crisp parametric bi-matrix games, we presented a method to find the (weak) Pareto Nash equilibrium strategy for fuzzy games. It is also easy to see that all the generalizations of Nash equilibrium strategy that we have defined here are natural extensions of the Nash equilibrium strategy for the crisp matrix games, and our models are more general than Maeda's models.

References

- [1] J.F. Nash, Non-cooperative games, *Annals of Mathematics* 54 (1951) 286–295.
- [2] D. Kahneman, A. Tversky, Prospect theory analysis of decision under risk, *Econometrica* 47 (1979) 263–292.
- [3] D. Butnariu, Fuzzy games: a description of time concept, *Fuzzy Sets and Systems* 1 (1978) 181–192.
- [4] L. Campos, Fuzzy linear programming model to solve fuzzy matrix game, *Fuzzy Sets and Systems* 32 (1989) 275–289.
- [5] R. Yager, A procedure for ordering fuzzy subsets of the unit interval, *Information Science* 24 (1981) 143–161.
- [6] I. Nishizaki, M. Sakawa, *Fuzzy and Multi-objective Games for Conflict Resolution*, Physica-verlag, New York, 2001.
- [7] I. Nishizaki, M. Sakawa, Equilibrium solutions in multi-objective bi-matrix games with fuzzy payoffs and fuzzy goals, *Fuzzy Sets and Systems* 111 (2000) 99–116.
- [8] C.R. Bector, S. Chandra, V. Vijay, Matrix games with fuzzy goals and fuzzy linear programming duality, *Fuzzy Optimization and Decision Making* 3 (2004) 255–269.
- [9] C.R. Bector, S. Chandra, *Fuzzy Mathematical Programming and Fuzzy Matrix Games*, Springer, Berlin, 2005.
- [10] V. Vijay, C.R. Bector, Bi-matrix games with fuzzy goals and fuzzy payoffs, *Fuzzy Optimization and Decision Making* 3 (2004) 327–344.
- [11] V. Vijay, S. Chandra, C.R. Bector, Matrix games with fuzzy goals and fuzzy payoffs, *Omega* 33 (2005) 425–429.
- [12] F. Kacher, M. Larbanib, Solution concept for a non-cooperative game with fuzzy parameters, *International Game Theory Review* 8 (2006) 489–498.
- [13] F. Kacher, M. Larbanib, Existence of equilibrium solution for a non-cooperative game with fuzzy goals and parameters, *Fuzzy Sets and Systems* 159 (2008) 164–176.
- [14] M. Larbani, Non cooperative fuzzy games in normal form: a survey, *Fuzzy Sets and Systems* 160 (2009) 3184–3210.
- [15] T. Maeda, On characterization of equilibrium strategy of two-person zero-sum games with fuzzy payoffs, *Fuzzy Sets and Systems* 139 (2003) 283–296.
- [16] T. Maeda, On characterization of equilibrium strategy of bi-matrix games with fuzzy payoffs, *Journal of Mathematical Analysis and Applications* 251 (2000) 885–896.
- [17] J. Ramík, J. Římanek, Inequality relation between fuzzy numbers and its use in fuzzy optimization, *Fuzzy Sets and Systems* 16 (1985) 123–150.
- [18] L.A. Zadeh, Fuzzy sets, *Information and Control* 8 (1965) 338–353.
- [19] L.A. Zadeh, Fuzzy sets as a basis for a theory of possibility, *Fuzzy Sets and Systems* 1 (1978) 3–28.