Int. J. Production Economics 118 (2009) 243–252

Contents lists available at ScienceDirect

Int. J. Production Economics

journal homepage: www.elsevier.com/locate/ijpe

Non-cooperative strategies for production and shipments lot sizing in the vendor–buyer system

Stanisław Bylka

Institute of Computer Science, Polish Academy of Sciences, 01-237 Warsaw, ul. Ordona 21, Poland

ARTICLE INFO

Available online 26 August 2008

Keywords:
Supply chain
Constrained game
Equilibrium strategies

ABSTRACT

This paper considers a decentralized dynamic production–distribution control. A discrete deterministic model in which a vendor produces a product and supplies it to the buyer is considered.

Several papers on vendor–buyer integrated production inventory management assume that policies are set by a central decision maker to optimize total system performance. Although vendor and buyer may agree to minimize the total cost, at least one of them has a private incentive to deviate from the agreement.

In the competitive situation, the objective is to determine schedules which minimize the individual average total cost of production, shipment and stockholding. We assume that the division of shipment costs is centrally coordinated or negotiated initially. It leads to a class of non-cooperative constrained games, indexed by two parameters connected with partitions of shipment costs. Non-cooperative strategies are considered as feasible strategies in a restricted non-cooperative game. Some properties of equilibrium strategies are investigated as acceptable equilibrium strategies of subgames in the game.

1. Introduction

One of the major tasks of supply chain management is to coordinate the processes in the supply chain in such a way that lowers system-wide cost is gained. In general, a supply chain is composed of independent partners with individual costs. For this reason, each firm is interested in minimizing its own cost independently. A well-integrated supply chain involves coordinating the flows of materials and information between distinct entities (as supplier, manufacturer, transporter, buyer, etc.). Both in the practice and in the literature considerable attention is paid to the importance of a coordinated relationships between entities in supply chain.

In the decentralized case, the power structure in vendor–buyer relationships as well as knowledge about the partner costs structure ought to be identified. Abad (1994) formulated the problem of buyer–vendor coordination as a two person cooperative game. Banerjee (1986a) and Goyal (1987) compute a price discount which compensates the loss of the buyer with respect to a cooperative policy. Kelle et al. (2003) also suggest some quantitative models to serve for motivation and a negotiating tool for providing joint operating policies. Sucky (2006) provides a bargaining model with asymmetric information in a dominated-buyer supply chain. He also describes the quantity losses due to a cooperative policy, both from the buyer's and the supplier's perspective. See also Viswanathan and Wang (2003) and Li et al. (2002) for leader–follower relationships under the Stackelberg game.

The idea of joint optimization for vendor and buyer was initiated by Goyal (1976) and Banerjee (1986b). A basic policy is any feasible policy where deliveries are made to the warehouse only when the warehouse has zero inventory. Several authors incorporated policies in

E-mail addresses: bylka@wars.ipipan.waw.pl, bylka@ipipan.waw.pl (S. Bylka).

0925-5273/$ - see front matter © 2008 Elsevier B.V. All rights reserved.
doi:10.1016/j.ijpe.2008.08.032

which sizes of successive shipments from the vendor to the buyer within a production cycle either increases by a factor equal to the ratio of production rate to the demand rate or are equal in size. Neither the equal shipment size policy nor the increasing shipment size policy is always optimal. Hill (1999) combining these two policies derived the structure of a globally optimal production and shipment policy. All policies mentioned above are members of the following class of (k,n)-policies: *Initial* $k\geqslant 0$ *subbatches increasing in sizes* (*the vendor dominates the buyer*) *precede* $n\geqslant 0$ *subbatches equal in sizes* (*the buyer dominates the vendor*). These policies can also be viewed from competition perspective—as non-cooperative strategies for production distribution system.

Supply chain management has recently received a great deal of attention in the economy. It is natural that potential savings in cooperation (in centralized case) cannot be ignored. Competitive pressures drive profitability down. It may force firms to reduce costs while maintaining excellent customer service. A comprehensive literature review of works in this field is presented in Douglas and Griffin (1996) and Sarmah et al. (2006). Most studies on game theory models of supply chain (see Huang and Li, 2001) consider agents which maximize individual profit functions (with respect to purchase and sale prices). Bylka (2003) investigated equilibrium strategies in each (k,n)-non-cooperative game under the assumption that only the division of shipment cost (given by the pair (k,n)) is central coordinated or negotiated initially—before the game.

In most paper dealing with integrated inventory models (not only mentioned above), the transportation cost is considered only as a part of fixed setup or ordering cost. Ertogral et al. (2007) have studied the effects of incorporation of transportation cost into the model on possibility for better decision making under equal size shipment policies ($(0,n)$-policies). A fundamental advance in the two-side cost structure is in recognizing how delivery-transportation costs apply to both sides. David and Eben-Chaime (2003) suggested such a separation for $(0,n)$-policies. However, there is an additional set of problems involved in implementing (k,n)-policies (strategies) used in the decentralized (competed) case. The main issues are whether and how the vendor participates in the transportation cost in the case $k>0$.

The research presented in this paper offers game model without prices, where agents minimize individual costs. It is a non-cooperative game model of vendor–buyer competition in terms of the number and size of batches transferred between the two sides. It is a generalization of the paper Bylka (2003) with respect to the assumptions on the class of (k,n)-strategies. Additionally, an aspect of stability of equilibrium strategies is considered. The remainder of the paper is organized as follows. In Section 2, we develop the model describing inventory patterns and cost structure under (k,n)-strategies. It is then assumed that the players (the vendor and the buyer) compete for their size of batch decisions through a (k,n)-game. Nash equilibrium strategies are constructed in Section 3. However, the competition concerns only the shipments with exogenous number of transferred batches. This restriction is relaxed in Section 4.

2. Modelling vendor–buyer relationships under $(\boldsymbol{k},\boldsymbol{n})$-strategies

We consider a continuous deterministic model of a production–distribution system for a single product. A vendor produces a product on a single machine and supplies it to the buyer. Buyer's demand is a continuous function of the time. We denote

P	production rate of the vendor;
D	demand rate of the buyer;
λ	ratio P/D.

We examine the situation, where a vendor produces a product in a batch production environment and supplies it to the buyer under deterministic conditions (see Goyal, 1976). A schedule is determined by a sequence of cycles, each of them determined by

Q	the size of production batch;
$m>0$	the number of shipments of subbatches per production cycle;
(q_j,t_j)	quantity and the moment of j-th shipment, $j=1,\dots,m$.

Specifically, the problem will be characterized by the following assumptions:

A1. Constant production rate is sufficient to meet buyer's demand ($\lambda>1$) and buyer's demand must be satisfied.
A2. The final product is distributed by shipping it in discrete lots from the vendor's stock to buyer's stock (realized instantaneously).

In each cycle of the size Q there are some shipments which replenish the buyer's stock. The production starts at the moment, say 0, when the buyer have some initial inventories. A schedule $\tilde{q}=[(q_1,t_1),\dots,(q_m,t_m)]$ determines the number, quantities and timing of shipments to the buyer. The production is stopped at the moment t^* and it starts again in the next cycle at T with the same buyer's initial inventories. We have

$$Q=\sum_{j=1}^{m}q_j,\quad T=\frac{Q}{D}\quad\text{and}\quad t^*=\frac{Q}{P}\left(=\frac{T}{\lambda}\right),\tag{1}$$

with the notation (for the vendor, $i=0$, and for the buyer, $i=1$):

$q_0=I_1(0)$	the buyer's initial inventory position;
$I_i(t)$	the inventory position at t just before the possible replenishment.

Therefore, for the initial cycle of the length T and $t\in[0,T]$, we have

$$I_0(t)=I_0(0)+P\min\{t,t^*\}-\sum\{q_j|t_j<t\},\quad j=1,\dots,m,$$

$$I_1(t) = I_0(0) - Dt + \sum\{q_j | t_j < t\}. \quad (2)$$

2.1. The class of (k, n)-strategies

We consider schedules with all identical cycles which satisfy same additional assumptions:

A3. The buyer receives shipments just to run out of the stock.

The above assumption has an inspiration in practice (the buyer inventory holding cost is larger compared to vendor inventory holding cost) and A3 is a natural assumption in joint buyer and vendor coordination models.

Also, one of the following properties were postulated in the literature:

4a. The vendor's inventory is zero just past each shipment. Shipments satisfy $q_{j+1} = \alpha q_j$ for a given $1 \leqslant \alpha \leqslant P/D = \lambda$ (Hill, 1997).
4b. A3 and, additionally, shipments are identical, i.e. $q_1 = \cdots = q_m$ (Lu, 1995).
4c. Shipments satisfy 4a and, additionally, $\alpha = \lambda$ (Goyal, 1995).

Remark 1. It is easy to see that a feasible schedule satisfies both A3 and 4a only if $\alpha = \lambda$. On the other hand, in one-stage models the individual optimal strategies (vendor's EOP and buyer's EOQ) satisfy 4a and A3, respectively.

We assume (see Hill, 1999; Bylka, 2003):

A4. In a schedule $\tilde{q} = [(q_1, t_1), \ldots, (q_m, t_m)]$, the number of shipments $m = k + n$, and we have

$$q_{j+1} = \lambda q_j \text{ for each } j < k \quad \text{and}$$
$$q_{k+1} = \cdots = q_{k+n} \quad \text{if only } n > 0.$$

Therefore, $t_k \leqslant t^*$ and, by assumption A4, the production batch is partitioned as $Q = Q^0 + M$, such that

$$Q^0 = \sum_{j=1}^{k} q_j \quad \text{and} \quad M = Q - Q^0.$$

The part Q^0 is transferred to the buyer in k (geometrically with respect to λ) increasing sizes—as postulates 4c. The rest of the size M is transferred (as postulates A3) in n subbatches of equal sizes. In Fig. 1 $k = 2$ and $n = 3$. We have

$$n = 0 \Longleftrightarrow M = 0 \quad \text{and} \quad k = 0 \Longleftrightarrow M = Q.$$

With respect to the assumptions A1–A4, we are looking for a "good" schedule as a production–distribution cycle. Each such a cycle is determined by two feasible pairs (q_0, Q) and (k, n), with $k + n > 0$. Namely, by assumption A4, we have

$$q_i = \begin{cases} \lambda^i q_0 & \text{for } i = 1, \ldots, k, \\ \dfrac{M}{n} & \text{for } i = k+1, \ldots, k+n, \end{cases} \qquad t_i = \frac{\sum_{j=0}^{i-1} q_j}{D}, \quad (3)$$

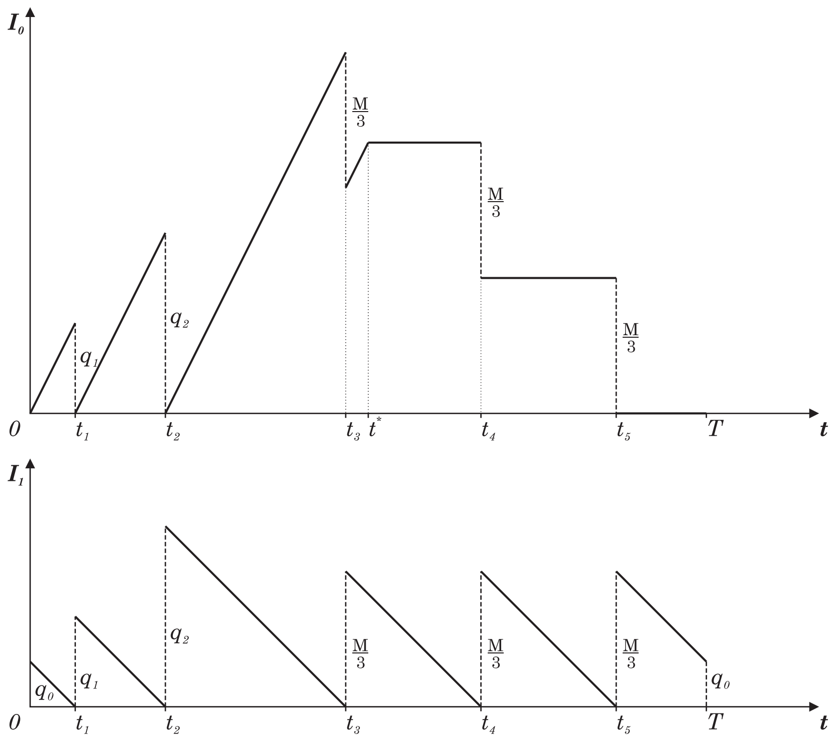

Fig. 1. The inventory positions for a schedule in a cycle, $M > 0$.

where

$$M = Q - \sum_{j=1}^{k} q_i \geqslant 0. \tag{4}$$

With respect to the feasibility, it ought to be

$$q_0 \frac{\lambda(\lambda^k - 1)}{\lambda - 1} = Q \quad \text{if only } n = 0.$$

In the case $n>0$, the feasibility means $q_0 \leqslant M/n$ and $\lambda q_k \geqslant M/n$, i.e.:

$$\frac{M}{\lambda n} \leqslant \lambda^k q_0 \leqslant \lambda^k \frac{M}{n} \quad \text{if only } n>0. \tag{5}$$

The set of all such feasible strategies will be denoted by Π.

Also, the schedule can be determined by two pairs $((Q,k),(M,n))$ if only there is a q_0 such that (3)–(5). From (4) it ought to be

$$Q - M = a_k q_0,$$

where

$$a_k = \frac{\lambda(\lambda^k - 1)}{\lambda - 1} \quad \text{for } k = 0, 1, \ldots . \tag{6}$$

It defines q_0 if only $k \geqslant 1$. If $k = 0$, then $q_1 = M/n$. Therefore, we can set

$$q_0 = \begin{cases} \dfrac{Q-M}{a_k} & \text{if } k \geqslant 1, \\ \dfrac{M}{\lambda n} & \text{if } k = 0. \end{cases} \tag{7}$$

Formally, we take

$$\Pi = \{((Q,k),(M,n)) | \text{there is } q_0 \text{ such that } (3)-(5)\}.$$

From now on, Π_k^n denotes the class of such strategies with given k and n, i.e.:

$$\Pi_k^n = \{(Q,M) | ((Q,k),(M,n)) \in \Pi\}. \tag{8}$$

We say also that such strategies are of (k,n)-type.[1] From strategies mentioned above only the class satisfying 4c (policies in Hill, 1997) is not a subclass of Π. Strategies satisfying 4a (see Goyal, 1976) and 4b (stationary policies in most cited paper) can be defined as members of Π_k^0 and Π_0^n, respectively.

2.2. Analytical consideration

In considered model, the vendor and the buyer cost parameters are the following:

A	fixed production set up cost;
h_i	unit stock holding cost for the vendor, $i = 0$, and for the buyer, $i = 1$;
A_i	transportation and replenishment cost per shipment, $i = 0, 1$.

In two-stage inventory models, it is assumed that $h_0 < h_1$. In our case it corresponds with the assumption A3.

[1] In this paper, it is more convenient that (k,n)-type means $(k+1,n)$-type in Bylka (2003) nomenclature.

Remark 2. For the cooperative case, policies of (k,n)-type have been inspired by practice and they are known in the literature. The average cost of the schedule $\tilde{q} = [(q_1,t_1),\ldots,(q_m,t_m)]$ on $[0,T]$ can be written as

$$C(\tilde{q}) = \frac{1}{T}\left[A + kA_0 + nA_1 + h_0 \int_0^T I_0(t)\,dt + h_1 \int_0^T I_1(t)\,dt\right],$$

where T and Q as in (1). Goyal (1976) and Lu (1995) found formulas for relatively optimal policies (optimal in the considered classes Π_k^0 and Π_0^n for cooperative case, respectively).

In the general case (for strategies from Π_k^n) it is important to know an additional number, say r, of *buyer's shipments in vendor's production time* $[0,t^*]$ (we have $r = 1$ and 0 in the cases in Figs. 1 and 2, respectively). More formally,

$$r \text{ is the natural number such that } t_{k+r} \leqslant t^* < t_{k+r+1}, \tag{9}$$

i.e. by (3) and (4) we have $r = 0 \Leftrightarrow \lambda^k q_0 > M/\lambda$ and

$$\frac{M}{\lambda} - r\frac{M}{n} < \lambda^k q_0 \leqslant \frac{M}{\lambda} - (r-1)\frac{M}{n} \quad \text{if only } r>0.$$

With respect to the constraints in (5), we have

$$0 \leqslant r \leqslant 1 + \frac{n-1}{\lambda}.$$

Remark 3. In cooperative case, Hill (1999) shows that optimal schedule $\tilde{q} = [(q_1,t_1),\ldots,(q_m,t_m)]$ is generated by a strategy from Π_k^n and, additionally, $r = 1$, i.e. it satisfies

$$\max\left\{\frac{M}{\lambda n}, \frac{M}{\lambda} - \frac{M}{n}\right\} < \lambda^k q_0 \leqslant \frac{M}{\lambda}. \tag{10}$$

With respect to the form of the functions $I_0(t)$ and $I_1(t)$ (see Figs. 1 and 2) it is easy to count their integrals. In the case $n = 0$, by (2), (3) and (7), we obtain

$$\int_0^T I_0(t)\,dt = \sum_{i=1}^{k} \frac{1}{2}(t_i - t_{i-1})q_i = \frac{\lambda q_0^2}{2D}\frac{\lambda^{2k} - 1}{\lambda^2 - 1} \tag{11}$$

and

$$\int_0^T I_1(t)\,dt = \sum_{i=1}^{k} \frac{1}{2}(t_{i+1} - t_i)q_i = \frac{\lambda^2 q_0^2}{2D}\,\frac{\lambda^{2k} - 1}{\lambda^2 - 1}. \tag{12}$$

In the other cases, it is easy to count the integrals for each r separately. For $r = 0$ we have

$$\begin{aligned} \int_0^T I_0(t)\,dt &= \sum_{i=1}^{k} \frac{1}{2}(t_i - t_{i-1})q_i + \frac{M}{2}\left(\frac{q_k}{D} + t_{k+1} - t^*\right) \\ &\quad + \frac{M^2}{D} - \sum_{i=1}^{n} \frac{iM}{nD}\frac{M}{n} \\ &= \frac{1}{2D}\left[q_0^2 \frac{\lambda(\lambda^{2k} - 1)}{\lambda^2 - 1} + 2M\lambda^k q_0 - \frac{M^2}{\lambda} + \frac{M^2(n-1)}{n}\right]. \end{aligned}$$

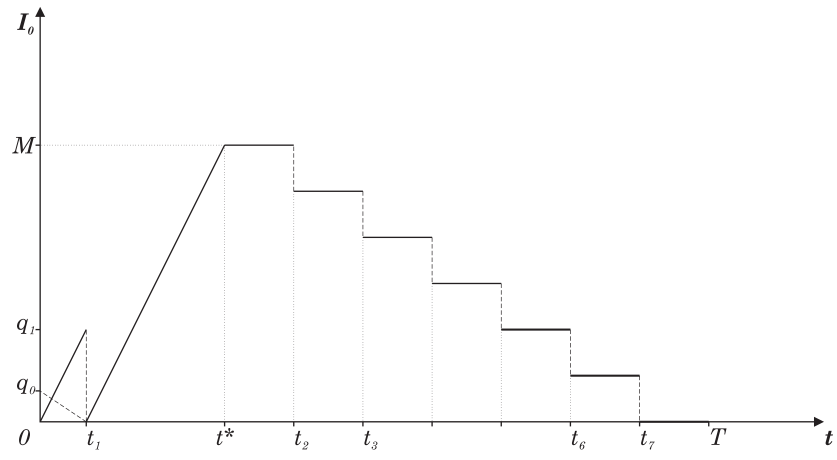

Fig. 2. The inventory positions of the vendor stock for a strategy from Π_1^6.

For $r \geqslant 1$ we have

$$
\begin{aligned}
\int_0^T I_0(t)\,\mathrm{d}t &= \sum_{i=1}^{k+1} \frac{1}{2}(t_i - t_{i-1})\lambda q_{i-1} \\
&\quad + \sum_{i=1}^{r-1} \frac{M}{nD}\frac{I_0(t_{k+i+1}) + I_0(t_{k+i}^{+})}{2} \\
&\quad + \sum_{i=1}^{n-r} \frac{iM}{nD}\frac{M}{n} - \frac{1}{2}(t^* - t_{k+r})^2 P \\
&= \frac{q_0^2}{2D}\frac{\lambda(\lambda^{2k+2}-1)}{\lambda^2-1} + \frac{M(r-1)}{2nD} \\
&\quad \times \left[2\lambda q_k + \frac{M}{n}(\lambda r - \lambda - r)\right] \\
&\quad + \frac{M^2(n-r)(n-r+1)}{2n^2 D} \\
&\quad - \frac{1}{2D}\left[\lambda q_k^2 + 2\lambda q_k M \frac{\lambda r - n - \lambda}{n} + \frac{M^2}{\lambda} \right. \\
&\quad \left. + \frac{M^2(r-1)(\lambda r - \lambda - 2n)}{n^2}\right] \\
&= \frac{1}{2D}\left[q_0^2 \frac{\lambda(\lambda^{2k}-1)}{\lambda^2-1} + 2M\lambda^k q_0 - \frac{M^2}{\lambda} + \frac{M^2(n-1)}{n}\right].
\end{aligned} \tag{13}
$$

Remark 4. It is worth to note that the expression of $\int_0^T I_0(t)\,\mathrm{d}t$ does not depend on r—the number of buyer's shipments in the vendor's production time.

For the buyer inventory we have

$$
\begin{aligned}
\int_0^T I_1(t)\,\mathrm{d}t &= \sum_{i=1}^{k} \frac{1}{2}(t_{i+1} - t_i)q_i + n\frac{1}{2}\frac{M}{nD}\frac{M}{n} = \sum_{i=1}^{k}\frac{q_i^2}{2D} + \frac{M^2}{2nD} \\
&= \frac{q_0^2}{2D}\frac{\lambda^2(\lambda^{2k}-1)}{\lambda^2-1} + \frac{M^2}{2nD}.
\end{aligned} \tag{14}
$$

2.3. Individual strategies in non-cooperative case

We define strategies of players which lead to a schedule with respect to a strategy from Π. Let us define the following constrained game:

G1. The vendor uses strategy (Q,k), where $Q>0$ is the size of production batch and $k \geqslant 0$ is the number of initial subbatches (controlled by the vendor) shipped with own cost. It also determines the length of the cycle $T = Q/D$.

G2. The buyer uses strategy (M,n), where $M \geqslant 0$ is the size of a lot which he will be shipped individually with own cost in $n \geqslant 0$ identical shipments. Additionally, $M=0$ if and only if $n>0$.

G3. A collection of strategies $s = ((Q,k),(M,n))$ is feasible if $s \in \Pi$. In addition, all model parameters are common knowledge.

In fact, a pair of strategies $((Q,k),(M,n))$ is feasible if and only if $Q \geqslant M$, because of Eq. (7). Every pair of strategies determines the quantity q_0 as well as a feasible schedule.

G4. For a feasible choice of strategies $s = ((Q,k),(M,n)) \in \Pi$ the vendor cost is equal to

$$V(s) = \frac{1}{T}\left[A + kA_0 + h_0 \int_0^T I_0(t)\,\mathrm{d}t\right] \tag{15}$$

and the buyer cost is equal to

$$B(s) = \frac{1}{T}\left[nA_1 + h_1 \int_0^T I_1(t)\,\mathrm{d}t\right]. \tag{16}$$

The constrained game defined by G1–G4 will be denoted by $\Gamma = (\Pi, V, B)$.

From now on we make the assumption that the agents decide about own participation in the transportation cost. They decide about the pair (k,n) before the game, for example, by negotiations. It leads to the suitable subgame of the game Γ—the new game $\Gamma_k^n = (\Pi_k^n, V_k^n, B_k^n)$—where a pair of strategies (Q, M) is feasible if it is an element of the set Π_k^n (given by Eq. (8)) and the costs

$$V_k^n(Q,M) = V((Q,k),(M,n)) \quad \text{and} \quad B_k^n(Q,M) = B((Q,k),(M,n)). \tag{17}$$

Definition 1. A pair of strategies $(Q^*, M^*) \in \Pi_k^n$ is called an equilibrium in Γ_k^n iff we have

$$V_k^n(Q^*,M^*) \leqslant V_k^n(Q,M^*) \quad \text{for every } (Q,M^*) \in \Pi_k^n$$

and

$$B_k^n(Q^*,M^*) \leqslant B_k^n(Q^*,M) \quad \text{for every } (Q^*,M) \in \Pi_k^n.$$

There are two natural questions:

- Is there an equilibrium in the game Γ_k^n?
- When a feasible pair of strategies in Γ_k^n (in particulary an equilibrium) is acceptable for the agents in Γ in the sense given below:

Definition 2. A collection of strategies $((Q,k),(M,n)) \in \Pi$ is acceptable for the vendor and for the buyer in the game Γ iff, respectively,

$$V((Q,k),(M,n)) \leqslant V((Q,k'),(M,n)) \quad \text{for every } ((Q,k'),(M,n)) \in \Pi$$

and

$$B((Q,k),(M,n)) \leqslant B((Q,k),(M,n')) \quad \text{for every } ((Q,k),(M,n')) \in \Pi.$$

Remark 5. The Γ_k^n games with the feasibility conditions given by $(Q,M) \in \Pi_k^n$ and, additionally, by (10) were investigated by Bylka (2003). In this paper a more natural case is considered. The mathematical condition (10) (see Remark 3 for more with respect to its interpretation) is relaxed.

3. Subgames equilibrium strategies

It is required in this section that a pair (k, n) is given. We shall now establish another expression for the constrained game $\Gamma_k^n = (\Pi_k^n, V_k^n, B_k^n)$.

The feasibility of (Q,M) means $M = Q$ iff $k = 0$ as well as $M = 0$ iff $n = 0$. In the other cases, with respect to (7), (5) can be expressed as

$$\frac{M}{\lambda^{k+1} n} \leqslant \frac{Q-M}{a_k} \leqslant \frac{M}{n} \quad \text{if } k>0 \text{ and } n>0. \tag{18}$$

It is a convex cone in $\mathscr{R}_+ \times \mathscr{R}_+$. Therefore, in the case $n>0$, the sets of possible responses of the vendor and the buyer on partner's strategies are the following closed intervals. For the vendor

$$v_k^n(M) = \left\{ Q \in \mathscr{R}_+ \middle| \left(1 + \frac{a_k}{n\lambda^{k+1}}\right) M \leqslant Q \leqslant \left(1 + \frac{a_k}{n}\right) M \right\} \tag{19}$$

and for the buyer

$$b_k^n(Q) = \left\{ M \in \mathscr{R}_+ \middle| \frac{n}{a_k + n} Q \leqslant M \leqslant \frac{n\lambda^{k+1}}{a_k + n\lambda^{k+1}} Q \right\}. \tag{20}$$

With respect to the cost functions, we obtain from (11), (12) and (7) that

$$\begin{aligned} V_k^0(Q,M) &= \frac{D}{Q}\left[A + kA_0 + h_0 \frac{\lambda(Q-M)^2}{2Da_k^2} \frac{\lambda^{2k}-1}{\lambda^2-1}\right] \\ &= \frac{1}{Q}\left[(A+kA_0)D + \frac{h_0}{2}\alpha_k (Q-M)^2\right] \quad \text{for } n = 0. \end{aligned} \tag{21}$$

For $n>0$, from (13), (15) and (7) we have

$$\begin{aligned} V_k^n(Q,M) &= \frac{D}{Q}\left\langle A + kA_0 + \frac{h_0}{2D}\left[\frac{(Q-M)^2}{a_k^2}\frac{\lambda(\lambda^{2k}-1)}{\lambda^2-1} \right.\right. \\ &\quad \left.\left. + 2M\lambda^k \frac{Q-M}{a_k} + \frac{M^2(n-1)}{n} - \frac{M^2}{\lambda}\right]\right\rangle \\ &= \frac{1}{Q}\left\langle (A+kA_0)D + \frac{h_0}{2}[\alpha_k(Q-M)^2 \right. \\ &\quad \left. + 2M\beta_k(Q-M) + \gamma_n M^2]\right\rangle, \end{aligned} \tag{22}$$

where

$$\alpha_k = \frac{\lambda^k + 1}{a_k(\lambda+1)}, \quad \beta_k = \frac{\lambda^k}{a_k} \quad \text{and} \quad \gamma_n = \frac{n-1}{n} - \frac{1}{\lambda}. \tag{23}$$

Analogously, the buyer's cost can be expressed as

$$\begin{aligned} B_k^0(Q,M) &= \frac{D}{Q} h_1 \frac{\lambda^2 (Q-M)^2}{2Da_k^2} \frac{\lambda^{2k}-1}{\lambda^2-1} \\ &= \frac{1}{Q}\frac{h_1}{2}\lambda\alpha_k(Q-M)^2 \quad \text{for } n = 0. \end{aligned} \tag{24}$$

For $n>0$, from (14), (16) and (7) we obtain

$$\begin{aligned} B_k^n(Q,M) &= \frac{D}{Q}\left\langle nA_1 + \frac{h_1}{2D}\left[\frac{(Q-M)^2}{a_k^2}\frac{\lambda^2(\lambda^{2k}-1)}{\lambda^2-1} + \frac{M^2}{n}\right]\right\rangle \\ &= \frac{1}{Q}\left\langle nA_1 D + \frac{h_1}{2}\left[\lambda\alpha_k(Q-M)^2 + \frac{1}{n}M^2\right]\right\rangle. \end{aligned} \tag{25}$$

We have the following result which concerns the equilibrium strategies.

Theorem 1. *For each pair of positive natural numbers (k, n) there exists a pure equilibrium in the game Γ_k^n.*[2] *If $k>0$ then such equilibrium is the one and only.*

Proof. For $k = 0$ the strategy $Q = M$ is the only one feasible answer of the vendor with respect to M and vice versa. Therefore each pair (M, M), $M \leqslant 0$, forms an equilibrium.

If $n = 0$ then $M = 0$ and every pair $(Q, 0)$ is feasible, because (5). The pair $(Q^*, 0)$, where Q^* satisfies

$$\frac{-(A+kA_0)D + \frac{h_0}{2}\alpha_k Q^{*2}}{Q^{*2}} = 0$$

is the unique equilibrium of Γ_k^0.

Assume $k>0$ and $n>0$. The buyer knows Q and he looks for the best possible response Q. For the cost (25) as a

[2] Let us remark that the proof of the analogous Theorem 1 (for more restricted constraints—see Remark 5) is given in Bylka (2003).

function of M we have

$$\frac{\partial B_k^n(Q,M)}{\partial M} = \frac{h_1}{Q}\left[\frac{M}{n} - \lambda\alpha_k(Q-M)\right].$$

The strategy

$$\mu_{\mathrm{b}}(Q) = \mu Q \quad \text{where } \mu = \frac{\lambda n\alpha_k}{1+\lambda n\alpha_k} \tag{26}$$

is the best response of the buyer on Q, if only the pair $(Q, \mu Q)$ is feasible.

The condition of feasibility, by (20), requires

$$\frac{n}{a_k+n}Q \leqslant \mu Q \leqslant \frac{n\lambda^{k+1}}{a_k+n\lambda^{k+1}}Q,$$

which is equivalent to

$$\frac{a_k}{\lambda^{k+1}n} \leqslant \frac{1}{\lambda n\alpha_k} \leqslant \frac{a_k}{n}.$$

It is easy to check, by (6) and (23), that the above inequalities are true in any case.

For the vendor's cost (22) as a function of Q we obtain

$$\frac{\partial V_k^n(Q,M)}{\partial Q} = \frac{\frac{h_0}{2}[\alpha_k Q^2 - M^2(\alpha_k - 2\beta_k + \gamma_n)] - (A+kA_0)D}{Q^2}.$$

We have

$$\frac{\partial V_k^n(Q,M)}{\partial Q} = 0 \Leftrightarrow z_k + y_k^n M^2 \geqslant 0 \quad \text{and} \quad Q = \sqrt{z_k + y_k^n M^2},$$

where

$$z_k = -\frac{2D(A+kA_k)}{h_0\alpha_k} \quad \text{and} \quad y_k^n = \frac{\alpha_k - 2\beta_k + \gamma_n}{\alpha_k}.$$

For the best response $\mu_{\mathrm{v}}(M)$ of the vendor on M, the condition of feasibility, by (19), requires

$$\left(1+\frac{a_k}{n\lambda^{k+1}}\right)M \leqslant \mu_{\mathrm{v}}(M) \leqslant \left(1+\frac{a_k}{n}\right)M,$$

which leads to

$$\mu_{\mathrm{v}}(M) = \begin{cases} \left(1+\frac{a_k}{n}\right)M & \text{if } M \leqslant \sqrt{\dfrac{z_k}{\left(1+\frac{a_k}{n}\right)^2 - y_k^n}}, \\ \sqrt{z_k + y_k^n M^2} & \text{if } \sqrt{\dfrac{z_k}{\left(1+\frac{a_k}{n}\right)^2 - y_k^n}} \leqslant M \\ & \leqslant \sqrt{\dfrac{z_k}{\left(1+\frac{a_k}{n\lambda^{k+1}}\right)^2 - y_k^n}}, \\ \left(1+\frac{a_k}{n\lambda^{k+1}}\right)M & \text{if } M \geqslant \sqrt{\dfrac{z_k}{\left(1+\frac{a_k}{n\lambda^{k+1}}\right)^2 - y_k^n}}. \end{cases} \tag{27}$$

We look for the vendor's strategy Q^* which is the best answer on the buyer's strategy $\mu_{\mathrm{b}}(Q^*)$. It can be found as a solution of the equation

$$Q = \mu_{\mathrm{v}}(\mu Q).$$

There is such a possibility if $Q = \sqrt{z_k + y_k^n \mu^2 Q^2}$. Therefore, Q^* can be found as a solution of the following equation:

$$\alpha_k Q^2 - \mu^2 Q(\alpha_k - 2\beta_k + \gamma_n) = \frac{2D(A+kA_0)}{h_0},$$

if only it has a positive solution. Indeed, the equivalence equation

$$Q^2\frac{\alpha_k(1+2\lambda n\alpha_k) + \lambda^2 n^2\alpha_k^2(2\beta_k - \gamma_n)}{(1+\lambda n\alpha_k)^2} = \frac{2D(A+kA_0)}{h_0}$$

has a positive solution because

$$\beta_k = \frac{\lambda^k}{a_k} = \frac{\lambda^k(\lambda-1)}{\lambda(\lambda^k - 1)} > \frac{\lambda-1}{\lambda} > \frac{n-1}{n} - \frac{1}{\lambda} = \gamma_n.$$

Therefore the pair (Q^*, M^*), where $M^* = \mu(Q^*)$ and

$$Q^* = \sqrt{\frac{2D(A+kA_0)(1+\lambda n\alpha_k)^2}{h_0[\alpha_k(1+2\lambda n\alpha_k) + \lambda^2 n^2\alpha_k^2(2\beta_k - \gamma_n)]}} \tag{28}$$

is the unique equilibrium. This is the desired conclusion. □

The proof above gives more, namely it is constructive because the explicit formulas (28) and (26) for Nash equilibrium strategies in each game Γ_k^n. Additionally, we have explicit formulas (26) and (27) for the best answers. It enables to consider the asymmetric positions in the Stackelberg game.

Remark 6. By (26), it is quite simple to find explicit formula for equilibrium pair $(Q^{\mathrm{s.v.}}, M^{\mathrm{s.v.}})$ in the case of the vendor as the leader and the buyer as the follower:

$$Q^{\mathrm{s.v.}} = \sqrt{\frac{2(A+kA_0)}{h_0[(1-\mu)^2\alpha_k + 2\mu(1-\mu)\beta_k - \mu^2\gamma_n]}} \quad \text{and}$$
$$M^{\mathrm{s.v.}} = \mu Q^{\mathrm{s.v.}}. \tag{29}$$

Calculation by (27) of the explicit formula for equilibrium pair $(Q^{\mathrm{s.b.}}, M^{\mathrm{s.b.}})$ in the case of the buyer as the leader and the vendor as the follower is too much complication.

4. Acceptability subgame's strategies in the game

We consider the problem of acceptability with respect to Definition 2. Let us first examine the sequence

$$\phi_i = \frac{\lambda^{i+1}}{a_i a_{i+1}}. \tag{30}$$

It is easy to see that the sequence ϕ_i is strictly decreasing.

Theorem 2. *Let $(Q,M) \in \Pi_k^n$ be a feasible pair of strategies in the subgame Γ_k^n. Then*

(i) *$((Q,k),(M,n))$ is acceptable for the vendor if and only if either $k=0$ or*

$$\Delta(Q,M) \equiv \frac{(Q-M)^2}{\lambda+1} + M(Q-M) \leqslant \frac{A_0 D}{h_0\phi_1} \quad \text{if } k = 1$$

and

$$\frac{A_0D}{h_0\phi_{k-1}} \leqslant \Delta(Q,M) \leqslant \frac{A_0D}{h_0\phi_k} \quad \text{if } k>1.$$

(ii) $((Q,k),(M,n))$ *is acceptable for the buyer if and only if either* $n=0$ *or*

$$\sqrt{\frac{2n(n-1)A_1D}{h_1}} \leqslant M \leqslant \sqrt{\frac{2n(n+1)A_1D}{h_1}} \quad \text{if } n\geqslant 1.$$

Proof. (ii) Our proof starts with the observation that $k=0$ if and only if $Q=M$.

Assume $k>0$. Using the form of the vendor's cost either (21) or (22), we have

$$\begin{aligned} &V_i^n(Q,M) - V_{i+1}^n(Q,M) \\ &= \frac{1}{Q}\left\langle (-A_0)D + \frac{h_0}{2}[(\alpha_i - \alpha_{i+1})(Q-M)^2 \right. \\ &\quad \left. +2M(\beta_i - \beta_{i+1})(Q-M)] \right\rangle \\ &= \frac{1}{Q}\left\langle -A_0D + \frac{h_0}{2}\left[\frac{2\phi_i}{\lambda+1}(Q-M)^2 + 2M(Q-M)\phi_i\right]\right\rangle \\ &= \frac{1}{Q}[-A_0D + h_0\Delta(Q,M)\phi_i]. \end{aligned}$$

Since ϕ_i is strictly decreasing, the vendor accepts $(Q,M) \in \Pi_k^n$ iff

$$V_k^n(Q,M) \leqslant V_{k+1}^n(Q,M) \ \Leftrightarrow\ h_0\Delta(Q,M)\phi_k \leqslant A_0D$$

and, if only $k>1$,

$$V_k^n(Q,M) \leqslant V_{k-1}^n(Q,M) \ \Leftrightarrow\ h_0\Delta(Q,M)\phi_{k-1} \geqslant A_0D.$$

This finishes the proof of (i).

(ii) Since $n=0$ if and only if $M=0$, every $((Q,k),(0,0))$ ought to be accepted by the buyer.

Assume $n>0$. We use similar arguments as in the proof of (i). Using the form of the buyer's cost either (24) or (25), we have

$$B_k^i(Q,M) - B_k^{i+1}(Q,M) = \frac{1}{Q}\left[-A_1D + \frac{h_1}{2}\frac{1}{i(i+1)}M^2\right].$$

We can continue in the same fashion as above in (i), because the sequence $1/i(i+1)$ is strictly decreasing. The buyer accepts $((Q,k),(M,n)) \in \Pi$ iff

$$A_1D \geqslant \frac{h_1}{2}\frac{1}{n(n+1)}M^2$$

and, if only $n>1$,

$$\frac{h_1}{2}\frac{1}{n(n+1)}M^2 \leqslant A_1D \leqslant \frac{h_1}{2}\frac{1}{n(n-1)}M^2,$$

which completed the proof of (ii) and also of the theorem. $\square$

Table 1

Equilibrium	q_0	t_1	q_1	t_2	q_2	t_3	q_3, q_4	t^*	r
Nash	20.81	0.021	66.59	0.087	213.07	0.30	178.2	0.199	0
Stackelberg leader b.	13.87	0.014	44.37	0.058	142.37	0.20	188.58	0.176	0
Stackelberg leader v.	22.57	0.023	72.22	0.095	231.1	0.33	193.27	0.225	0
Nash Bylka (2003)	11.69	0.011	37.42	0.049	119.72	0.169	191.54	0.169	1
Optimal Hill (1999)	7.38	0.007	23.64	0.031	75.63	0.107	229.27	0.74	1

Table 2

Equilibrium	Q	M	$V_2^2(Q,M)$	$B_2^2(Q,M)$	$V_2^2+B_2^2$	v.acc.	b.acc.
Nash	636.06	356.39	1308.9	524.10	1833.00	Yes	No
Stackelberg leader b.	563.53	377.16	1237.89	502.44	1800.33	Yes	No
Stackelberg leader v.	689.87	386.55	1304.59	555.66	1860.26	No	No
Nash Bylka (2003)	540.21	383.09	1292.65	504.94	1797.59	Yes	No
Optimal Hill (1999)	557.8	458.54	1203.8	588.95	1792.76	Yes	No

Table 3

Equilibrium	Q	M	$V_2^4(Q,M)$	$B_2^4(Q,M)$	$V_2^4+B_2^4$	v.acc.	b.acc.
Nash	585.31	420.38	1412.52	433.59	1846.11	Yes	Yes
Stackelberg leader b.	568.49	425.53	1410.72	432.25	1842.97	Yes	Yes
Stackelberg leader v.	639.67	459.42	1406.97	443.47	1850.44	Yes	No
Nash Bylka (2003)	585.31	420.38	1412.52	433.59	1846.11	Yes	Yes

5. Numerical example

Goyal introduced a numerical example to illustrate his solution procedure. We use his example to present our solution procedure for equilibrium strategies. The total cost found by the method in this study is compared with the costs found in Hill (1999) and Bylka (2003), where the same numerical example was tested.

The data for the example are

$$A = 400, \quad A_1 = A_2 = 25, \quad h_1 = 4, \quad h_2 = 5,$$

$$D = 1000, \quad P = 3200.$$

For this example we determine (k,n)-equilibrium strategies for each pair with $k = 2$ and $n = 2$ or 4. The equilibrium strategies in the game Γ_2^2 are presented in Tables 1 and 2. The schedule under equilibrium in more restricted analogous game in Bylka (2003) and the schedule under optimal with respect to system cost strategies in Hill (1999) are added.

The individual and system costs, under schedules presented in Table 1 are given in Table 2. The answer with respect to acceptability is given in two last columns of Table 2. Let us remark that each pair of strategies is not acceptable for the buyer. The number of shipments controlled by the buyer in the subgame Γ_2^2 is too small for him. It is diametrically opposed to the case of the subgame Γ_2^4 presented in Table 3.

6. Conclusions

In strong competition, vendor and buyer look for equilibrium strategies in a non-cooperative game. It is well known that joint optimal strategy will always result in savings in total system cost. However, the implementation of such a strategy requires coordination and cooperation. While each vendor and buyer incurs own inventory holding cost, the transportation (or, more general, delivery associated) costs were charged on both partners. Even if the system contains a mechanism which completely recognize individual costs, the join optimal strategy may be questioned. Specially, that it requires concordance with respect to the arbitrary indicated partition of the total system cost. Is an agreement possible? Maybe? This agreement is easier if such partitions are more possible.

In this paper we propose a two progressive coordination model in supply management. The first step—an agreement with respect to the class of strategies. The second step—restricted non-cooperation game, where both number and size of batches transferred between the two partners are decision variables.

A mathematical analysis of (k,n)-policies for integrated vendor–buyer inventory system (with respect to their possible competition) was presented. Such policies were used as strategies in (k,n)-subgames of a non-cooperative constrained game. Formulas for Nash and Stackelberg equilibrium strategies in subgames were presented. The problem of existence of equilibrium in the main game stays open. In this matter, we considered only the problem of acceptability of subgames equilibrium strategies in the main game.

As a subject for further research, it is worth to analyze this model in a more precise scenario with respect to delivery associated costs.

References

Abad, P.L., 1994. Supplier pricing and lot sizing when demand is price sensitive. European Journal of Operational Research 78, 334–354.
Banerjee, 1986a. A quantity discount pricing model to increase vendor profits. Management Science 32, 1513–1517.
Banerjee, 1986b. A joint economic-lot-size model for purchaser and vendor. Decision Sciences 17, 292–311.
Bylka, S., 2003. Competitive and cooperative policies for the vendor–buyer system. International Journal of Production Economics 81–82, 533–544.
David, I., Eben-Chaime, M., 2003. How far should JIT vendor–buyer relationships go? International Journal of Production Economics 81–82, 361–368.
Douglas, J.T., Griffin, P.M., 1996. Coordinated supply chain management. European Journal of Operational Research 94, 1–15.
Ertogral, K.M., Darwish, M., Ben-Daya, M., 2007. Production and shipment lot sizing in a vendor–buyer supply chain with transportation cost. European Journal of Operational Research 176, 1592–1606.
Goyal, S.K., 1976. An integrated inventory model for a single supplier single customer system. International Journal of Production Research 14, 107–111.
Goyal, S.K., 1987. Comment on: A generalized quantity discount pricing model to increase supplier's profits. Management Science 33, 1635–1636.
Goyal, S.K., 1995. A one vendor multi-buyer integrated inventory model: A comment. European Journal of Operational Research 81, 209–210.
Hill, R.M., 1999. The optimal production and shipment policy for the single-vendor single-buyer integrated production inventory problem. European Journal of Operational Research 37, 2463–2475.
Hill, R.M., 1997. The single-vendor single-buyer integrated production-inventory model with a generalised policy. European Journal of Operational Research 97, 493–499.
Huang, Z., Li, S.X., 2001. Co-op advertising models in manufacturer–retailer supply chains: A game theory approach. European Journal of Operational Research 135, 527–544.
Kelle, P., Al-khateeb, F., Miller, P.A., 2003. Partnership and negotiation support by joint optimal ordering/setup policies for JIT. International Journal of Production Economics 81–82, 431–441.

Li, S.X., Hung, Z., Zhu, J., Chan, P.Y.K., 2002. Cooperative advertising, game theory and manufacturer–retailer supply chain. Omega 30, 347–357.
Lu, L., 1995. A one-vendor multi-buyer integrated inventory model. European Journal of Operational Research 81, 312–324.
Sarmah, S.P., Acharya, D., Goyal, S.K., 2006. Buyer vendor coordinated models in supply chain management. European Journal of Operational Research 175, 1–15.
Sucky, E., 2006. A bargaining model with asymmetric information for a single supplier–single buyer problem. European Journal of Operational Research 171, 516–535.
Viswanathan, S., Wang, Q., 2003. Discount pricing decisions in distribution channels with price-sensitive demand. European Journal of Operational Research 149, 571–587.