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# Non-cooperative strategies for production and shipments lot sizing in the vendor–buyer system

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## ABSTRACT

This paper considers a decentralized dynamic production–distribution control. A discrete deterministic model in which a vendor produces a product and supplies it to the buyer is considered.

Several papers on vendor–buyer integrated production inventory management assume that policies are set by a central decision maker to optimize total system performance. Although vendor and buyer may agree to minimize the total cost, at least one of them has a private incentive to deviate from the agreement.

In the competitive situation, the objective is to determine schedules which minimize the individual average total cost of production, shipment and stockholding. We assume that the division of shipment costs is centrally coordinated or negotiated initially. It leads to a class of non-cooperative constrained games, indexed by two parameters connected with partitions of shipment costs. Non-cooperative strategies are considered as feasible strategies in a restricted non-cooperative game. Some properties of equilibrium strategies are investigated as acceptable equilibrium strategies of subgames in the game.

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## 1. Introduction

One of the major tasks of supply chain management is to coordinate the processes in the supply chain in such a way that lowers system-wide cost is gained. In general, a supply chain is composed of independent partners with individual costs. For this reason, each firm is interested in minimizing its own cost independently. A well-integrated supply chain involves coordinating the flows of materials and information between distinct entities (as supplier, manufacturer, transporter, buyer, etc.). Both in the practice and in the literature considerable attention is paid to the importance of a coordinated relationships between entities in supply chain.

In the decentralized case, the power structure in vendor–buyer relationships as well as knowledge about

the partner costs structure ought to be identified. [Abad \(1994\)](#) formulated the problem of buyer–vendor coordination as a two person cooperative game. [Banerjee \(1986a\)](#) and [Goyal \(1987\)](#) compute a price discount which compensates the loss of the buyer with respect to a cooperative policy. [Kelle et al. \(2003\)](#) also suggest some quantitative models to serve for motivation and a negotiating tool for providing joint operating policies. [Sucky \(2006\)](#) provides a bargaining model with asymmetric information in a dominated–buyer supply chain. He also describes the quantity losses due to a cooperative policy, both from the buyer's and the supplier's perspective. See also [Viswanathan and Wang \(2003\)](#) and [Li et al. \(2002\)](#) for leader–follower relationships under the Stackelberg game.

The idea of joint optimization for vendor and buyer was initiated by [Goyal \(1976\)](#) and [Banerjee \(1986b\)](#). A basic policy is any feasible policy where deliveries are made to the warehouse only when the warehouse has zero inventory. Several authors incorporated policies in

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which sizes of successive shipments from the vendor to the buyer within a production cycle either increases by a factor equal to the ratio of production rate to the demand rate or are equal in size. Neither the equal shipment size policy nor the increasing shipment size policy is always optimal. Hill (1999) combining these two policies derived the structure of a globally optimal production and shipment policy. All policies mentioned above are members of the following class of  $(k, n)$ -policies: *Initial  $k \geq 0$  subbatches increasing in sizes (the vendor dominates the buyer) precede  $n \geq 0$  subbatches equal in sizes (the buyer dominates the vendor)*. These policies can also be viewed from competition perspective—as non-cooperative strategies for production distribution system.

Supply chain management has recently received a great deal of attention in the economy. It is natural that potential savings in cooperation (in centralized case) cannot be ignored. Competitive pressures drive profitability down. It may force firms to reduce costs while maintaining excellent customer service. A comprehensive literature review of works in this field is presented in Douglas and Griffin (1996) and Sarmah et al. (2006). Most studies on game theory models of supply chain (see Huang and Li, 2001) consider agents which maximize individual profit functions (with respect to purchase and sale prices). Bylka (2003) investigated equilibrium strategies in each  $(k, n)$ -non-cooperative game under the assumption that only the division of shipment cost (given by the pair  $(k, n)$ ) is central coordinated or negotiated initially—before the game.

In most paper dealing with integrated inventory models (not only mentioned above), the transportation cost is considered only as a part of fixed setup or ordering cost. Ertogral et al. (2007) have studied the effects of incorporation of transportation cost into the model on possibility for better decision making under equal size shipment policies ( $(0, n)$ -policies). A fundamental advance in the two-side cost structure is in recognizing how delivery-transportation costs apply to both sides. David and Eben-Chaime (2003) suggested such a separation for  $(0, n)$ -policies. However, there is an additional set of problems involved in implementing  $(k, n)$ -policies (strategies) used in the decentralized (competed) case. The main issues are whether and how the vendor participates in the transportation cost in the case  $k > 0$ .

The research presented in this paper offers game model without prices, where agents minimize individual costs. It is a non-cooperative game model of vendor–buyer competition in terms of the number and size of batches transferred between the two sides. It is a generalization of the paper Bylka (2003) with respect to the assumptions on the class of  $(k, n)$ -strategies. Additionally, an aspect of stability of equilibrium strategies is considered. The remainder of the paper is organized as follows. In Section 2, we develop the model describing inventory patterns and cost structure under  $(k, n)$ -strategies. It is then assumed that the players (the vendor and the buyer) compete for their size of batch decisions through a  $(k, n)$ -game. Nash equilibrium strategies are constructed in Section 3. However, the competition concerns only the

shipments with exogenous number of transferred batches. This restriction is relaxed in Section 4.

## 2. Modelling vendor–buyer relationships under $(k, n)$ -strategies

We consider a continuous deterministic model of a production–distribution system for a single product. A vendor produces a product on a single machine and supplies it to the buyer. Buyer’s demand is a continuous function of the time. We denote

- $P$  production rate of the vendor;
- $D$  demand rate of the buyer;
- $\lambda$  ratio  $P/D$ .

We examine the situation, where a vendor produces a product in a batch production environment and supplies it to the buyer under deterministic conditions (see Goyal, 1976). A schedule is determined by a sequence of cycles, each of them determined by

- $Q$  the size of production batch;
- $m > 0$  the number of shipments of subbatches per production cycle;
- $(q_j, t_j)$  quantity and the moment of  $j$ -th shipment,  $j = 1, \dots, m$ .

Specifically, the problem will be characterized by the following assumptions:

- A1. Constant production rate is sufficient to meet buyer’s demand ( $\lambda > 1$ ) and buyer’s demand must be satisfied.
- A2. The final product is distributed by shipping it in discrete lots from the vendor’s stock to buyer’s stock (realized instantaneously).

In each cycle of the size  $Q$  there are some shipments which replenish the buyer’s stock. The production starts at the moment, say 0, when the buyer have some initial inventories. A schedule  $\bar{q} = [(q_1, t_1), \dots, (q_m, t_m)]$  determines the number, quantities and timing of shipments to the buyer. The production is stopped at the moment  $t^*$  and it starts again in the next cycle at  $T$  with the same buyer’s initial inventories. We have

$$Q = \sum_{j=1}^m q_j, \quad T = \frac{Q}{D} \quad \text{and} \quad t^* = \frac{Q}{P} \left( = \frac{T}{\lambda} \right), \quad (1)$$

with the notation (for the vendor,  $i = 0$ , and for the buyer,  $i = 1$ ):

- $q_0 = I_1(0)$  the buyer’s initial inventory position;
- $I_i(t)$  the inventory position at  $t$  just before the possible replenishment.

Therefore, for the initial cycle of the length  $T$  and  $t \in [0, T]$ , we have

$$I_0(t) = I_0(0) + P \min\{t, t^*\} - \sum_{j=1}^m \{q_j | t_j < t\}, \quad j = 1, \dots, m,$$

$$I_1(t) = I_0(0) - Dt + \sum \{q_j | t_j < t\}. \tag{2}$$

(vendor's EOP and buyer's EOQ) satisfy 4a and A3, respectively.

2.1. The class of (k, n)-strategies

We consider schedules with all identical cycles which satisfy same additional assumptions:

A3. The buyer receives shipments just to run out of the stock.

The above assumption has an inspiration in practice (the buyer inventory holding cost is larger compared to vendor inventory holding cost) and A3 is a natural assumption in joint buyer and vendor coordination models.

Also, one of the following properties were postulated in the literature:

- 4a. The vendor's inventory is zero just past each shipment. Shipments satisfy  $q_{j+1} = \alpha q_j$  for a given  $1 \leq \alpha \leq P/D = \lambda$  (Hill, 1997).
- 4b. A3 and, additionally, shipments are identical, i.e.  $q_1 = \dots = q_m$  (Lu, 1995).
- 4c. Shipments satisfy 4a and, additionally,  $\alpha = \lambda$  (Goyal, 1995).

**Remark 1.** It is easy to see that a feasible schedule satisfies both A3 and 4a only if  $\alpha = \lambda$ . On the other hand, in one-stage models the individual optimal strategies

We assume (see Hill, 1999; Bylka, 2003):

A4. In a schedule  $\tilde{q} = [(q_1, t_1), \dots, (q_m, t_m)]$ , the number of shipments  $m = k + n$ , and we have

$$q_{j+1} = \lambda q_j \text{ for each } j < k \text{ and } q_{k+1} = \dots = q_{k+n} \text{ if only } n > 0.$$

Therefore,  $t_k \leq t^*$  and, by assumption A4, the production batch is partitioned as  $Q = Q^0 + M$ , such that

$$Q^0 = \sum_{j=1}^k q_j \text{ and } M = Q - Q^0.$$

The part  $Q^0$  is transferred to the buyer in  $k$  (geometrically with respect to  $\lambda$ ) increasing sizes—as postulates 4c. The rest of the size  $M$  is transferred (as postulates A3) in  $n$  subbatches of equal sizes. In Fig. 1  $k = 2$  and  $n = 3$ . We have

$$n = 0 \iff M = 0 \text{ and } k = 0 \iff M = Q.$$

With respect to the assumptions A1–A4, we are looking for a “good” schedule as a production–distribution cycle. Each such a cycle is determined by two feasible pairs  $(q_0, Q)$  and  $(k, n)$ , with  $k + n > 0$ . Namely, by assumption A4, we have

$$q_i = \begin{cases} \lambda^i q_0 & \text{for } i = 1, \dots, k, \\ \frac{M}{n} & \text{for } i = k + 1, \dots, k + n, \end{cases} \quad t_i = \frac{\sum_{j=0}^{i-1} q_j}{D}, \tag{3}$$

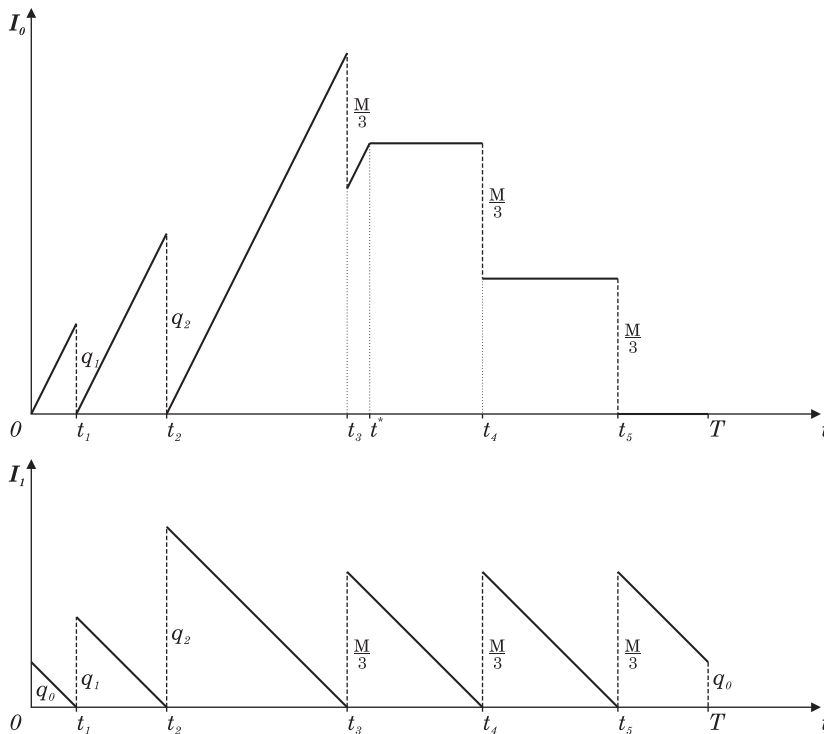


Fig. 1. The inventory positions for a schedule in a cycle,  $M > 0$ .

where

$$M = Q - \sum_{j=1}^k q_j \geq 0. \tag{4}$$

With respect to the feasibility, it ought to be

$$q_0 \frac{\lambda(\lambda^k - 1)}{\lambda - 1} = Q \quad \text{if only } n = 0.$$

In the case  $n > 0$ , the feasibility means  $q_0 \leq M/n$  and  $\lambda q_k \geq M/n$ , i.e.:

$$\frac{M}{\lambda n} \leq \lambda^k q_0 \leq \lambda^k \frac{M}{n} \quad \text{if only } n > 0. \tag{5}$$

The set of all such feasible strategies will be denoted by  $\Pi$ .

Also, the schedule can be determined by two pairs  $((Q, k), (M, n))$  if only there is a  $q_0$  such that (3)–(5). From (4) it ought to be

$$Q - M = a_k q_0,$$

where

$$a_k = \frac{\lambda(\lambda^k - 1)}{\lambda - 1} \quad \text{for } k = 0, 1, \dots \tag{6}$$

It defines  $q_0$  if only  $k \geq 1$ . If  $k = 0$ , then  $q_1 = M/n$ . Therefore, we can set

$$q_0 = \begin{cases} \frac{Q - M}{a_k} & \text{if } k \geq 1, \\ \frac{M}{\lambda n} & \text{if } k = 0. \end{cases} \tag{7}$$

Formally, we take

$$\Pi = \{((Q, k), (M, n)) \mid \text{there is } q_0 \text{ such that (3)–(5)}\}.$$

From now on,  $\Pi_k^n$  denotes the class of such strategies with given  $k$  and  $n$ , i.e.:

$$\Pi_k^n = \{(Q, M) \mid ((Q, k), (M, n)) \in \Pi\}. \tag{8}$$

We say also that such strategies are of  $(k, n)$ -type.<sup>1</sup> From strategies mentioned above only the class satisfying 4c (policies in Hill, 1997) is not a subclass of  $\Pi$ . Strategies satisfying 4a (see Goyal, 1976) and 4b (stationary policies in most cited paper) can be defined as members of  $\Pi_k^0$  and  $\Pi_0^n$ , respectively.

2.2. Analytical consideration

In considered model, the vendor and the buyer cost parameters are the following:

- A fixed production set up cost;
- $h_i$  unit stock holding cost for the vendor,  $i = 0$ , and for the buyer,  $i = 1$ ;
- $A_i$  transportation and replenishment cost per shipment,  $i = 0, 1$ .

In two-stage inventory models, it is assumed that  $h_0 < h_1$ . In our case it corresponds with the assumption A3.

**Remark 2.** For the cooperative case, policies of  $(k, n)$ -type have been inspired by practice and they are known in the literature. The average cost of the schedule  $\tilde{q} = [(q_1, t_1), \dots, (q_m, t_m)]$  on  $[0, T]$  can be written as

$$C(\tilde{q}) = \frac{1}{T} \left[ A + kA_0 + nA_1 + h_0 \int_0^T I_0(t) dt + h_1 \int_0^T I_1(t) dt \right],$$

where  $T$  and  $Q$  as in (1). Goyal (1976) and Lu (1995) found formulas for relatively optimal policies (optimal in the considered classes  $\Pi_k^0$  and  $\Pi_0^n$  for cooperative case, respectively).

In the general case (for strategies from  $\Pi_k^n$ ) it is important to know an additional number, say  $r$ , of buyer's shipments in vendor's production time  $[0, t^*]$  (we have  $r = 1$  and 0 in the cases in Figs. 1 and 2, respectively). More formally,

$$r \text{ is the natural number such that } t_{k+r} \leq t^* < t_{k+r+1}, \tag{9}$$

i.e. by (3) and (4) we have  $r = 0 \Leftrightarrow \lambda^k q_0 > M/\lambda$  and

$$\frac{M}{\lambda} - r \frac{M}{n} < \lambda^k q_0 \leq \frac{M}{\lambda} - (r - 1) \frac{M}{n} \quad \text{if only } r > 0.$$

With respect to the constraints in (5), we have

$$0 \leq r \leq 1 + \frac{n - 1}{\lambda}.$$

**Remark 3.** In cooperative case, Hill (1999) shows that optimal schedule  $\tilde{q} = [(q_1, t_1), \dots, (q_m, t_m)]$  is generated by a strategy from  $\Pi_k^n$  and, additionally,  $r = 1$ , i.e. it satisfies

$$\max \left\{ \frac{M}{\lambda n}, \frac{M}{\lambda} - \frac{M}{n} \right\} < \lambda^k q_0 \leq \frac{M}{\lambda}. \tag{10}$$

With respect to the form of the functions  $I_0(t)$  and  $I_1(t)$  (see Figs. 1 and 2) it is easy to count their integrals. In the case  $n = 0$ , by (2), (3) and (7), we obtain

$$\int_0^T I_0(t) dt = \sum_{i=1}^k \frac{1}{2} (t_i - t_{i-1}) q_i = \frac{\lambda q_0^2 \lambda^{2k} - 1}{2D \lambda^2 - 1} \tag{11}$$

and

$$\int_0^T I_1(t) dt = \sum_{i=1}^k \frac{1}{2} (t_{i+1} - t_i) q_i = \frac{\lambda^2 q_0^2 \lambda^{2k} - 1}{2D \lambda^2 - 1}. \tag{12}$$

In the other cases, it is easy to count the integrals for each  $r$  separately. For  $r = 0$  we have

$$\begin{aligned} \int_0^T I_0(t) dt &= \sum_{i=1}^k \frac{1}{2} (t_i - t_{i-1}) q_i + \frac{M}{2} \left( \frac{q_k}{D} + t_{k+1} - t^* \right) \\ &\quad + \frac{M^2}{D} - \sum_{i=1}^n \frac{iM}{nD} \\ &= \frac{1}{2D} \left[ q_0^2 \frac{\lambda(\lambda^{2k} - 1)}{\lambda^2 - 1} + 2M\lambda^k q_0 - \frac{M^2}{\lambda} + \frac{M^2(n - 1)}{n} \right]. \end{aligned}$$

<sup>1</sup> In this paper, it is more convenient that  $(k, n)$ -type means  $(k + 1, n)$ -type in Bylka (2003) nomenclature.

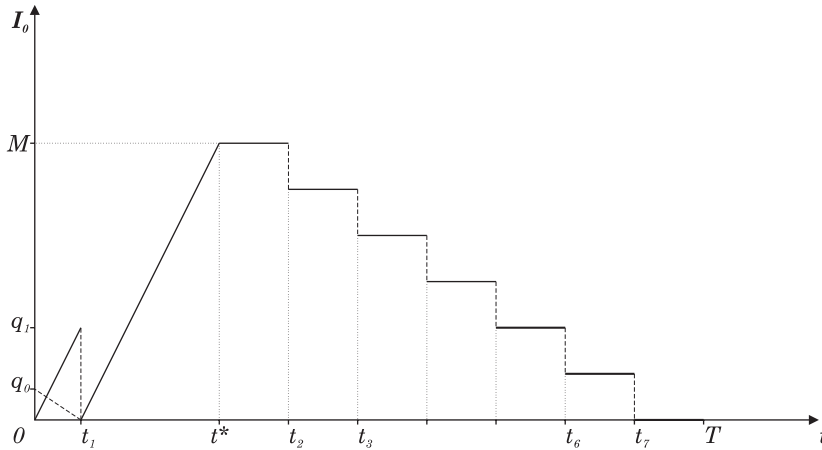


Fig. 2. The inventory positions of the vendor stock for a strategy from  $\Pi_1^6$ .

For  $r \geq 1$  we have

$$\begin{aligned} \int_0^T I_0(t) dt &= \sum_{i=1}^{k+1} \frac{1}{2} (t_i - t_{i-1}) \lambda q_{i-1} \\ &+ \sum_{i=1}^{r-1} \frac{M}{nD} \frac{I_0(t_{k+i+1}) + I_0(t_{k+i}^+)}{2} \\ &+ \sum_{i=1}^{n-r} \frac{iM}{nD} \frac{M}{n} - \frac{1}{2} (t^* - t_{k+r})^2 P \\ &= \frac{q_0^2}{2D} \frac{\lambda(\lambda^{2k+2} - 1)}{\lambda^2 - 1} + \frac{M(r-1)}{2nD} \\ &\times \left[ 2\lambda q_k + \frac{M}{n} (\lambda r - \lambda - r) \right] \\ &+ \frac{M^2(n-r)(n-r+1)}{2n^2D} \\ &- \frac{1}{2D} \left[ \lambda q_k^2 + 2\lambda q_k M \frac{\lambda r - n - \lambda}{n} + \frac{M^2}{\lambda} \right. \\ &\left. + \frac{M^2(r-1)(\lambda r - \lambda - 2n)}{n^2} \right] \\ &= \frac{1}{2D} \left[ q_0^2 \frac{\lambda(\lambda^{2k} - 1)}{\lambda^2 - 1} + 2M\lambda^k q_0 - \frac{M^2}{\lambda} + \frac{M^2(n-1)}{n} \right]. \end{aligned} \tag{13}$$

**Remark 4.** It is worth to note that the expression of  $\int_0^T I_0(t) dt$  does not depend on  $r$ —the number of buyer's shipments in the vendor's production time.

For the buyer inventory we have

$$\begin{aligned} \int_0^T I_1(t) dt &= \sum_{i=1}^k \frac{1}{2} (t_{i+1} - t_i) q_i + n \frac{1}{2} \frac{M}{nD} \frac{M}{n} = \sum_{i=1}^k \frac{q_i^2}{2D} + \frac{M^2}{2nD} \\ &= \frac{q_0^2}{2D} \frac{\lambda^2(\lambda^{2k} - 1)}{\lambda^2 - 1} + \frac{M^2}{2nD}. \end{aligned} \tag{14}$$

### 2.3. Individual strategies in non-cooperative case

We define strategies of players which lead to a schedule with respect to a strategy from  $\Pi$ . Let us define the following constrained game:

- G1. The vendor uses strategy  $(Q, k)$ , where  $Q > 0$  is the size of production batch and  $k \geq 0$  is the number of initial subbatches (controlled by the vendor) shipped with own cost. It also determines the length of the cycle  $T = Q/D$ .
- G2. The buyer uses strategy  $(M, n)$ , where  $M \geq 0$  is the size of a lot which he will be shipped individually with own cost in  $n \geq 0$  identical shipments. Additionally,  $M = 0$  if and only if  $n > 0$ .
- G3. A collection of strategies  $s = ((Q, k), (M, n))$  is feasible if  $s \in \Pi$ . In addition, all model parameters are common knowledge.

In fact, a pair of strategies  $((Q, k), (M, n))$  is feasible if and only if  $Q \geq M$ , because of Eq. (7). Every pair of strategies determines the quantity  $q_0$  as well as a feasible schedule.

- G4. For a feasible choice of strategies  $s = ((Q, k), (M, n)) \in \Pi$  the vendor cost is equal to

$$V(s) = \frac{1}{T} \left[ A + kA_0 + h_0 \int_0^T I_0(t) dt \right] \tag{15}$$

and the buyer cost is equal to

$$B(s) = \frac{1}{T} \left[ nA_1 + h_1 \int_0^T I_1(t) dt \right]. \tag{16}$$

The constrained game defined by G1–G4 will be denoted by  $\Gamma = (\Pi, V, B)$ .

From now on we make the assumption that the agents decide about own participation in the transportation cost. They decide about the pair  $(k, n)$  before the game, for example, by negotiations. It leads to the suitable subgame

of the game  $\Gamma$ —the new game  $\Gamma_k^n = (\Pi_k^n, V_k^n, B_k^n)$ —where a pair of strategies  $(Q, M)$  is feasible if it is an element of the set  $\Pi_k^n$  (given by Eq. (8)) and the costs

$$V_k^n(Q, M) = V((Q, k), (M, n)) \quad \text{and} \quad B_k^n(Q, M) = B((Q, k), (M, n)). \tag{17}$$

**Definition 1.** A pair of strategies  $(Q^*, M^*) \in \Pi_k^n$  is called an equilibrium in  $\Gamma_k^n$  iff we have

$$V_k^n(Q^*, M^*) \leq V_k^n(Q, M^*) \quad \text{for every } (Q, M^*) \in \Pi_k^n$$

and

$$B_k^n(Q^*, M^*) \leq B_k^n(Q^*, M) \quad \text{for every } (Q^*, M) \in \Pi_k^n.$$

There are two natural questions:

- Is there an equilibrium in the game  $\Gamma_k^n$ ?
- When a feasible pair of strategies in  $\Gamma_k^n$  (in particular an equilibrium) is acceptable for the agents in  $\Gamma$  in the sense given below:

**Definition 2.** A collection of strategies  $((Q, k), (M, n)) \in \Pi$  is acceptable for the vendor and for the buyer in the game  $\Gamma$  iff, respectively,

$$V((Q, k), (M, n)) \leq V((Q, k'), (M, n))$$

for every  $((Q, k'), (M, n)) \in \Pi$

and

$$B((Q, k), (M, n)) \leq B((Q, k), (M, n'))$$

for every  $((Q, k), (M, n')) \in \Pi$ .

**Remark 5.** The  $\Gamma_k^n$  games with the feasibility conditions given by  $(Q, M) \in \Pi_k^n$  and, additionally, by (10) were investigated by Bylka (2003). In this paper a more natural case is considered. The mathematical condition (10) (see Remark 3 for more with respect to its interpretation) is relaxed.

### 3. Subgames equilibrium strategies

It is required in this section that a pair  $(k, n)$  is given. We shall now establish another expression for the constrained game  $\Gamma_k^n = (\Pi_k^n, V_k^n, B_k^n)$ .

The feasibility of  $(Q, M)$  means  $M = Q$  iff  $k = 0$  as well as  $M = 0$  iff  $n = 0$ . In the other cases, with respect to (7), (5) can be expressed as

$$\frac{M}{\lambda^{k+1}n} \leq \frac{Q-M}{a_k} \leq \frac{M}{n} \quad \text{if } k > 0 \text{ and } n > 0. \tag{18}$$

It is a convex cone in  $\mathcal{R}_+ \times \mathcal{R}_+$ . Therefore, in the case  $n > 0$ , the sets of possible responses of the vendor and the buyer on partner's strategies are the following closed intervals. For the vendor

$$v_k^n(M) = \left\{ Q \in \mathcal{R}_+ \mid \left(1 + \frac{a_k}{n\lambda^{k+1}}\right)M \leq Q \leq \left(1 + \frac{a_k}{n}\right)M \right\} \tag{19}$$

and for the buyer

$$b_k^n(Q) = \left\{ M \in \mathcal{R}_+ \mid \frac{n}{a_k + n}Q \leq M \leq \frac{n\lambda^{k+1}}{a_k + n\lambda^{k+1}}Q \right\}. \tag{20}$$

With respect to the cost functions, we obtain from (11), (12) and (7) that

$$\begin{aligned} V_k^n(Q, M) &= \frac{D}{Q} \left[ A + kA_0 + h_0 \frac{\lambda(Q-M)^2 \lambda^{2k} - 1}{2Da_k^2 \lambda^2 - 1} \right] \\ &= \frac{1}{Q} \left[ (A + kA_0)D + \frac{h_0}{2} \alpha_k(Q-M)^2 \right] \quad \text{for } n = 0. \end{aligned} \tag{21}$$

For  $n > 0$ , from (13), (15) and (7) we have

$$\begin{aligned} V_k^n(Q, M) &= \frac{D}{Q} \left\langle A + kA_0 + \frac{h_0}{2D} \left[ \frac{(Q-M)^2 \lambda(\lambda^{2k} - 1)}{a_k^2 \lambda^2 - 1} \right. \right. \\ &\quad \left. \left. + 2M\lambda^k \frac{Q-M}{a_k} + \frac{M^2(n-1)}{n} - \frac{M^2}{\lambda} \right] \right\rangle \\ &= \frac{1}{Q} \left\langle (A + kA_0)D + \frac{h_0}{2} [\alpha_k(Q-M)^2 \right. \\ &\quad \left. + 2M\beta_k(Q-M) + \gamma_n M^2] \right\rangle, \end{aligned} \tag{22}$$

where

$$\alpha_k = \frac{\lambda^k + 1}{a_k(\lambda + 1)}, \quad \beta_k = \frac{\lambda^k}{a_k} \quad \text{and} \quad \gamma_n = \frac{n-1}{n} - \frac{1}{\lambda}. \tag{23}$$

Analogously, the buyer's cost can be expressed as

$$\begin{aligned} B_k^n(Q, M) &= \frac{D}{Q} h_1 \frac{\lambda^2(Q-M)^2 \lambda^{2k} - 1}{2Da_k^2 \lambda^2 - 1} \\ &= \frac{1}{Q} \frac{h_1}{2} \lambda \alpha_k (Q-M)^2 \quad \text{for } n = 0. \end{aligned} \tag{24}$$

For  $n > 0$ , from (14), (16) and (7) we obtain

$$\begin{aligned} B_k^n(Q, M) &= \frac{D}{Q} \left\langle nA_1 + \frac{h_1}{2D} \left[ \frac{(Q-M)^2 \lambda^2 (\lambda^{2k} - 1)}{a_k^2 \lambda^2 - 1} + \frac{M^2}{n} \right] \right\rangle \\ &= \frac{1}{Q} \left\langle nA_1 D + \frac{h_1}{2} \left[ \lambda \alpha_k (Q-M)^2 + \frac{1}{n} M^2 \right] \right\rangle. \end{aligned} \tag{25}$$

We have the following result which concerns the equilibrium strategies.

**Theorem 1.** For each pair of positive natural numbers  $(k, n)$  there exists a pure equilibrium in the game  $\Gamma_k^n$ .<sup>2</sup> If  $k > 0$  then such equilibrium is the one and only.

**Proof.** For  $k = 0$  the strategy  $Q = M$  is the only one feasible answer of the vendor with respect to  $M$  and vice versa. Therefore each pair  $(M, M)$ ,  $M \leq 0$ , forms an equilibrium.

If  $n = 0$  then  $M = 0$  and every pair  $(Q, 0)$  is feasible, because (5). The pair  $(Q^*, 0)$ , where  $Q^*$  satisfies

$$\frac{-(A + kA_0)D + \frac{h_0}{2} \alpha_k Q^{*2}}{Q^{*2}} = 0$$

is the unique equilibrium of  $\Gamma_k^n$ .

Assume  $k > 0$  and  $n > 0$ . The buyer knows  $Q$  and he looks for the best possible response  $Q$ . For the cost (25) as a

<sup>2</sup> Let us remark that the proof of the analogous Theorem 1 (for more restricted constraints—see Remark 5) is given in Bylka (2003).

function of  $M$  we have

$$\frac{\partial B_k^n(Q, M)}{\partial M} = \frac{h_1}{Q} \left[ \frac{M}{n} - \lambda \alpha_k(Q - M) \right].$$

The strategy

$$\mu_b(Q) = \mu Q \quad \text{where } \mu = \frac{\lambda n \alpha_k}{1 + \lambda n \alpha_k} \quad (26)$$

is the best response of the buyer on  $Q$ , if only the pair  $(Q, \mu Q)$  is feasible.

The condition of feasibility, by (20), requires

$$\frac{n}{a_k + n} Q \leq \mu Q \leq \frac{n \lambda^{k+1}}{a_k + n \lambda^{k+1}} Q,$$

which is equivalent to

$$\frac{a_k}{\lambda^{k+1} n} \leq \frac{1}{\lambda n \alpha_k} \leq \frac{a_k}{n}.$$

It is easy to check, by (6) and (23), that the above inequalities are true in any case.

For the vendor's cost (22) as a function of  $Q$  we obtain

$$\frac{\partial V_k^n(Q, M)}{\partial Q} = \frac{h_0 [\alpha_k Q^2 - M^2 (\alpha_k - 2\beta_k + \gamma_n)] - (A + kA_0)D}{Q^2}.$$

We have

$$\frac{\partial V_k^n(Q, M)}{\partial Q} = 0 \Leftrightarrow z_k + y_k^n M^2 \geq 0 \quad \text{and} \quad Q = \sqrt{z_k + y_k^n M^2},$$

where

$$z_k = -\frac{2D(A + kA_0)}{h_0 \alpha_k} \quad \text{and} \quad y_k^n = \frac{\alpha_k - 2\beta_k + \gamma_n}{\alpha_k}.$$

For the best response  $\mu_v(M)$  of the vendor on  $M$ , the condition of feasibility, by (19), requires

$$\left(1 + \frac{a_k}{n \lambda^{k+1}}\right) M \leq \mu_v(M) \leq \left(1 + \frac{a_k}{n}\right) M,$$

which leads to

$$\mu_v(M) = \begin{cases} \left(1 + \frac{a_k}{n}\right) M & \text{if } M \leq \sqrt{\frac{z_k}{\left(1 + \frac{a_k}{n}\right)^2 - y_k^n}}, \\ \sqrt{z_k + y_k^n M^2} & \text{if } \sqrt{\frac{z_k}{\left(1 + \frac{a_k}{n}\right)^2 - y_k^n}} \leq M \\ & \leq \sqrt{\frac{z_k}{\left(1 + \frac{a_k}{n \lambda^{k+1}}\right)^2 - y_k^n}}, \\ \left(1 + \frac{a_k}{n \lambda^{k+1}}\right) M & \text{if } M \geq \sqrt{\frac{z_k}{\left(1 + \frac{a_k}{n \lambda^{k+1}}\right)^2 - y_k^n}}. \end{cases} \quad (27)$$

We look for the vendor's strategy  $Q^*$  which is the best answer on the buyer's strategy  $\mu_b(Q^*)$ . It can be found as a solution of the equation

$$Q = \mu_v(\mu Q).$$

There is such a possibility if  $Q = \sqrt{z_k + y_k^n \mu^2 Q^2}$ . Therefore,  $Q^*$  can be found as a solution of the following equation:

$$\alpha_k Q^2 - \mu^2 Q (\alpha_k - 2\beta_k + \gamma_n) = \frac{2D(A + kA_0)}{h_0},$$

if only it has a positive solution. Indeed, the equivalence equation

$$Q^2 \frac{\alpha_k (1 + 2\lambda n \alpha_k) + \lambda^2 n^2 \alpha_k^2 (2\beta_k - \gamma_n)}{(1 + \lambda n \alpha_k)^2} = \frac{2D(A + kA_0)}{h_0}$$

has a positive solution because

$$\beta_k = \frac{\lambda^k}{a_k} = \frac{\lambda^k (\lambda - 1)}{\lambda (\lambda^k - 1)} > \frac{\lambda - 1}{\lambda} > \frac{n - 1}{n} - \frac{1}{\lambda} = \gamma_n.$$

Therefore the pair  $(Q^*, M^*)$ , where  $M^* = \mu(Q^*)$  and

$$Q^* = \sqrt{\frac{2D(A + kA_0)(1 + \lambda n \alpha_k)^2}{h_0 [\alpha_k (1 + 2\lambda n \alpha_k) + \lambda^2 n^2 \alpha_k^2 (2\beta_k - \gamma_n)]}} \quad (28)$$

is the unique equilibrium. This is the desired conclusion.  $\square$

The proof above gives more, namely it is constructive because the explicit formulas (28) and (26) for Nash equilibrium strategies in each game  $\Gamma_k^n$ . Additionally, we have explicit formulas (26) and (27) for the best answers. It enables to consider the asymmetric positions in the Stackelberg game.

**Remark 6.** By (26), it is quite simple to find explicit formula for equilibrium pair  $(Q^{s.v.}, M^{s.v.})$  in the case of the vendor as the leader and the buyer as the follower:

$$Q^{s.v.} = \sqrt{\frac{2(A + kA_0)}{h_0 [(1 - \mu)^2 \alpha_k + 2\mu(1 - \mu)\beta_k - \mu^2 \gamma_n]}} \quad \text{and} \\ M^{s.v.} = \mu Q^{s.v.} \quad (29)$$

Calculation by (27) of the explicit formula for equilibrium pair  $(Q^{s.b.}, M^{s.b.})$  in the case of the buyer as the leader and the vendor as the follower is too much complication.

#### 4. Acceptability subgame's strategies in the game

We consider the problem of acceptability with respect to Definition 2. Let us first examine the sequence

$$\phi_i = \frac{\lambda^{i+1}}{a_i a_{i+1}}. \quad (30)$$

It is easy to see that the sequence  $\phi_i$  is strictly decreasing.

**Theorem 2.** Let  $(Q, M) \in \Pi_k^n$  be a feasible pair of strategies in the subgame  $\Gamma_k^n$ . Then

- (i)  $((Q, k), (M, n))$  is acceptable for the vendor if and only if either  $k = 0$  or

$$\Delta(Q, M) \equiv \frac{(Q - M)^2}{\lambda + 1} + M(Q - M) \leq \frac{A_0 D}{h_0 \phi_1} \quad \text{if } k = 1$$

and

$$\frac{A_0 D}{h_0 \phi_{k-1}} \leq \Delta(Q, M) \leq \frac{A_0 D}{h_0 \phi_k} \quad \text{if } k > 1.$$

(ii)  $((Q, k), (M, n))$  is acceptable for the buyer if and only if either  $n = 0$  or

$$\sqrt{\frac{2n(n-1)A_1 D}{h_1}} \leq M \leq \sqrt{\frac{2n(n+1)A_1 D}{h_1}} \quad \text{if } n \geq 1.$$

**Proof.** (ii) Our proof starts with the observation that  $k = 0$  if and only if  $Q = M$ .

Assume  $k > 0$ . Using the form of the vendor's cost either (21) or (22), we have

$$\begin{aligned} V_i^n(Q, M) - V_{i+1}^n(Q, M) &= \frac{1}{Q} \left\langle (-A_0)D + \frac{h_0}{2} [(\alpha_i - \alpha_{i+1})(Q - M)^2 + 2M(\beta_i - \beta_{i+1})(Q - M)] \right\rangle \\ &= \frac{1}{Q} \left\langle -A_0 D + \frac{h_0}{2} \left[ \frac{2\phi_i}{\lambda + 1} (Q - M)^2 + 2M(Q - M)\phi_i \right] \right\rangle \\ &= \frac{1}{Q} [-A_0 D + h_0 \Delta(Q, M)\phi_i]. \end{aligned}$$

Since  $\phi_i$  is strictly decreasing, the vendor accepts  $(Q, M) \in \Pi_k^n$  iff

$$V_k^n(Q, M) \leq V_{k+1}^n(Q, M) \Leftrightarrow h_0 \Delta(Q, M)\phi_k \leq A_0 D$$

and, if only  $k > 1$ ,

$$V_k^n(Q, M) \leq V_{k-1}^n(Q, M) \Leftrightarrow h_0 \Delta(Q, M)\phi_{k-1} \geq A_0 D.$$

This finishes the proof of (i).

(ii) Since  $n = 0$  if and only if  $M = 0$ , every  $((Q, k), (0, 0))$  ought to be accepted by the buyer.

Assume  $n > 0$ . We use similar arguments as in the proof of (i). Using the form of the buyer's cost either (24) or (25), we have

$$B_k^i(Q, M) - B_k^{i+1}(Q, M) = \frac{1}{Q} \left[ -A_1 D + \frac{h_1}{2} \frac{1}{i(i+1)} M^2 \right].$$

We can continue in the same fashion as above in (i), because the sequence  $1/i(i+1)$  is strictly decreasing. The buyer accepts  $((Q, k), (M, n)) \in \Pi$  iff

$$A_1 D \geq \frac{h_1}{2} \frac{1}{n(n+1)} M^2$$

and, if only  $n > 1$ ,

$$\frac{h_1}{2} \frac{1}{n(n+1)} M^2 \leq A_1 D \leq \frac{h_1}{2} \frac{1}{n(n-1)} M^2,$$

which completed the proof of (ii) and also of the theorem.  $\square$

**Table 1**

| Equilibrium           | $q_0$ | $t_1$ | $q_1$ | $t_2$ | $q_2$  | $t_3$ | $q_3, q_4$ | $t^*$ | $r$ |
|-----------------------|-------|-------|-------|-------|--------|-------|------------|-------|-----|
| Nash                  | 20.81 | 0.021 | 66.59 | 0.087 | 213.07 | 0.30  | 178.2      | 0.199 | 0   |
| Stackelberg leader b. | 13.87 | 0.014 | 44.37 | 0.058 | 142.37 | 0.20  | 188.58     | 0.176 | 0   |
| Stackelberg leader v. | 22.57 | 0.023 | 72.22 | 0.095 | 231.1  | 0.33  | 193.27     | 0.225 | 0   |
| Nash                  |       |       |       |       |        |       |            |       |     |
| Bylka (2003)          | 11.69 | 0.011 | 37.42 | 0.049 | 119.72 | 0.169 | 191.54     | 0.169 | 1   |
| Optimal               |       |       |       |       |        |       |            |       |     |
| Hill (1999)           | 7.38  | 0.007 | 23.64 | 0.031 | 75.63  | 0.107 | 229.27     | 0.74  | 1   |

**Table 2**

| Equilibrium           | $Q$    | $M$    | $V_2^2(Q, M)$ | $B_2^2(Q, M)$ | $V_2^2 + B_2^2$ | v.acc. | b.acc. |
|-----------------------|--------|--------|---------------|---------------|-----------------|--------|--------|
| Nash                  | 636.06 | 356.39 | 1308.9        | 524.10        | 1833.00         | Yes    | No     |
| Stackelberg leader b. | 563.53 | 377.16 | 1237.89       | 502.44        | 1800.33         | Yes    | No     |
| Stackelberg leader v. | 689.87 | 386.55 | 1304.59       | 555.66        | 1860.26         | No     | No     |
| Nash                  |        |        |               |               |                 |        |        |
| Bylka (2003)          | 540.21 | 383.09 | 1292.65       | 504.94        | 1797.59         | Yes    | No     |
| Optimal               |        |        |               |               |                 |        |        |
| Hill (1999)           | 557.8  | 458.54 | 1203.8        | 588.95        | 1792.76         | Yes    | No     |



Table 3

| Equilibrium           | Q      | M      | $V_2^4(Q, M)$ | $B_2^4(Q, M)$ | $V_2^4 + B_2^4$ | v.acc. | b.acc. |
|-----------------------|--------|--------|---------------|---------------|-----------------|--------|--------|
| Nash                  | 585.31 | 420.38 | 1412.52       | 433.59        | 1846.11         | Yes    | Yes    |
| Stackelberg leader b. | 568.49 | 425.53 | 1410.72       | 432.25        | 1842.97         | Yes    | Yes    |
| Stackelberg leader v. | 639.67 | 459.42 | 1406.97       | 443.47        | 1850.44         | Yes    | No     |
| Nash Bylka (2003)     | 585.31 | 420.38 | 1412.52       | 433.59        | 1846.11         | Yes    | Yes    |

## 5. Numerical example

Goyal introduced a numerical example to illustrate his solution procedure. We use his example to present our solution procedure for equilibrium strategies. The total cost found by the method in this study is compared with the costs found in Hill (1999) and Bylka (2003), where the same numerical example was tested.

The data for the example are

$$A = 400, \quad A_1 = A_2 = 25, \quad h_1 = 4, \quad h_2 = 5,$$

$$D = 1000, \quad P = 3200.$$

For this example we determine  $(k, n)$ -equilibrium strategies for each pair with  $k = 2$  and  $n = 2$  or  $4$ . The equilibrium strategies in the game  $\Gamma_2^2$  are presented in Tables 1 and 2. The schedule under equilibrium in more restricted analogous game in Bylka (2003) and the schedule under optimal with respect to system cost strategies in Hill (1999) are added.

The individual and system costs, under schedules presented in Table 1 are given in Table 2. The answer with respect to acceptability is given in two last columns of Table 2. Let us remark that each pair of strategies is not acceptable for the buyer. The number of shipments controlled by the buyer in the subgame  $\Gamma_2^2$  is too small for him. It is diametrically opposed to the case of the subgame  $\Gamma_2^4$  presented in Table 3.

## 6. Conclusions

In strong competition, vendor and buyer look for equilibrium strategies in a non-cooperative game. It is well known that joint optimal strategy will always result in savings in total system cost. However, the implementation of such a strategy requires coordination and cooperation. While each vendor and buyer incurs own inventory holding cost, the transportation (or, more general, delivery associated) costs were charged on both partners. Even if the system contains a mechanism which completely recognize individual costs, the joint optimal strategy may be questioned. Specially, that it requires concordance with respect to the arbitrary indicated partition of the total system cost. Is an agreement possible? Maybe? This agreement is easier if such partitions are more possible.

In this paper we propose a two progressive coordination model in supply management. The first step—an agreement with respect to the class of strategies. The

second step—restricted non-cooperation game, where both number and size of batches transferred between the two partners are decision variables.

A mathematical analysis of  $(k, n)$ -policies for integrated vendor–buyer inventory system (with respect to their possible competition) was presented. Such policies were used as strategies in  $(k, n)$ -subgames of a non-cooperative constrained game. Formulas for Nash and Stackelberg equilibrium strategies in subgames were presented. The problem of existence of equilibrium in the main game stays open. In this matter, we considered only the problem of acceptability of subgames equilibrium strategies in the main game.

As a subject for further research, it is worth to analyze this model in a more precise scenario with respect to delivery associated costs.

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