Int. J. Production Economics 143 (2013) 424–433

Contents lists available at SciVerse ScienceDirect

Int. J. Production Economics

journal homepage: www.elsevier.com/locate/ijpe

Non-cooperative consignment stock strategies for management in supply chain

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ARTICLE INFO

Available online 19 August 2011

Keywords:
Supply chain
Inventory
Consignment stock
Constrained game

ABSTRACT

This paper analyzes the coordination and competition issue in a two-level supply chain, having one vendor (or manufacturer) and one buyer (or retailer). A continuous deterministic model is presented. To satisfy the buyer's demands, the product is delivered in discrete batches from the vendor's stock to the buyer's stock and all shipments are realized instantaneously. We describe inventory patterns and the cost structure of production–distribution cycles (PDC) under generalized consignment stock (CS) policies. For the joint optimization case, the average total cost of production, shipment and stock-holding is minimized. Optimal solution techniques are presented and illustrated with numerical examples.

In a competitive situation, the objective is to determine schedules, which minimize the individual average total costs in the PDC obtainable by individual decisions. This paper presents a non-cooperative two-person constrained game with agents (a vendor and a buyer) choosing the number and sizes of deliveries. Generalized CS-policies are considered as feasible individual strategies in the game. We consider the class of non-cooperative sub-games, indexed by two integer parameters connected with CS policies. It is proven that there exists a unique Nash equilibrium strategy in each of the considered sub-game.

1. Introduction

Supply chain management has recently received a great attention in economics. In general, a supply chain is composed of independent partners with individual costs. When applied to productive environments, it allows (for isolated situations) the vendor to calculate the Economic Production Quantity (EPQ), although it might be significantly different from the buyer's EOQ. One of the major task of management is to coordinate the processes of the supply chain in such a way that the lowest system-wide cost is gained.

The idea of joint optimization for vendor and buyer was initiated by Goyal (1976), Banerjee (1986) and Lu (1995). Several authors (see the literature review by Sarmah et al. (2006)) incorporated policies in which the sizes of successive shipments (from the vendor to the buyer within a production cycle) either are equal in size or increases by a factor equal to the ratio of production rate to the demand rate. Hill (1999) combining these two ideas shows that in an optimal cycle the total production is transferred in deliveries of initially increasing and then equal size. However, there is an additional set of problems involved in implementing policies (strategies) with respect to whether and how the agents participate in the delivery-transportation costs.

Some researchers suggest quantitative models to describe the motivation and negotiating tools for providing joint operating policies. There is a lot of research done dealing with problems of coordinating a distribution system under vendor-managed inventory and consignment arrangements (Gümüs et al., 2008; Chen et al., 2010; Ru and Wang, 2010). An analytic formulation of consignment stock (CS) is discussed in Braglia and Zavanella (2003) and Zanoni and Grubbström (2004). In the case of CS-policies there are three decisions to take: about the delivered amount, about the number of deliveries making up the production batch, and how many deliveries should be delayed once this amount is available for shipment. For the case of multiple buyers, see Zavanella and Zanoni (2009) where an analytic formulation of consignment stock policies is given as well as the exhaustive references to this subject.

Generalized consignment stock (k,n)-policy (say CS(k,n)-policy) requires that:

- initial $k \geq 0$ deliveries are identical in sizes (say, of the size q^v),
- preceded $n \geq 0$ deliveries are also equal in sizes, say q^b.

Zanoni and Grubbström (2004) consider the case where $q^v = q^b$. Some relations for individual costs of inventories (in the

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0925-5273/$ - see front matter © 2012 Published by Elsevier B.V.
doi:10.1016/j.ijpe.2011.07.018

case when the agents can make decisions according to their own preferences) were analyzed by Gümüs et al. (2008).

In this paper we assume that the delayed deliveries are dispatched to the buyer's stock as late as it is possible instead of previously assumed "as soon as possible". For the central coordination case, such policy has lower cost only if the unit stockholding cost increase as stock moves down the supply chain. An analytic formulation of the modified CS policy is presented and an implicit solution is given.

The second part of this paper presents a game-theoretic approach for the case with non-equal sizes of deliveries. A model is considered, where agents minimize their individual costs under the assumption that only the division of shipment costs is coordinated centrally or negotiated. The CS(k,n)-policy can be viewed, from competition perspective, as a pair of non-cooperative vendor–buyer strategies, analogously as Hill's policies in Bylka (2009).

The paper is organized as follows. In Section 2, we develop the model describing inventory patterns under (k,n)-consignment stock policies. The costs of optimal policies in the considered classes CS(k,n)-policies are given in Sections 3 and 4. Formulas for Nash equilibrium strategies in the restricted non-cooperative games are presented in Section 5.

2. Modeling vendor–buyer relationships under CS(k,n)-policies

We consider a continuous deterministic model of a production–distribution system for a single product. A vendor produces a product on a single machine and supplies it to the buyer. Buyer's demand is represented as a continuous function of the time. We denote by P the vendor's production rate, by D the buyer's demand rate and the ratio $\lambda = P/D$.

We examine the situation, where a vendor produces a product in a batch production environment and supplies it to the buyer under deterministic conditions (see Goyal, 1976). A schedule is determined by a sequence of production–distribution cycles, each of them determined by the following quantities:

Q=the size of production batch;
$m>0$=the number of shipments per production–distribution cycle;
(q_j,t_j)=quantity and the moment of j-th shipment, $j=1,\ldots,m$.

The problem, in question, can be characterized as follows.

A1. Constant production rate is sufficient to meet buyer's demand ($\lambda>1$) and buyer's demand must be satisfied.
A2. The final product is distributed by shipping it in discrete lots from the vendor's stock to buyer's stock (realized instantaneously).
A3. The vendor's stock becomes empty just past each of the first set of (uniform size) deliveries, while the remaining (uniform size) deliveries are sent as late as it is possible. *This assumption will be commented below.*

In the considered model, the vendor and the buyer cost parameters are the following:

A=fixed production set up cost;
h_i=unit stock holding cost for the vendor, i=0, and for the buyer i=1;
A_i=delivery (transportation and replenishment) cost per shipment, i=0,1.

The objective is to determine a production and shipment schedule which minimizes the average (joint or buyer–vendor individual) total cost of production, shipment and stockholding. A policy ought to define a mode of decision making which determined a schedule. Policy can be given as a pair of strategies (vendor's and buyer's decision functions).

2.1. The production–distribution cycle

In each production–distribution cycle (PDC) the production starts at a moment, say 0, when the buyer have some initial inventories q_0. A schedule

$$\tilde{q} = [(q_1,t_1),\ldots,(q_m,t_m)] \quad \text{where} \sum_{j=0}^{i-1} \frac{q_j}{D} \ge t_i, \quad i=1,\ldots m$$

determines the number, quantities and timing of successive shipments to the buyer. The production is stopped at the moment t^* and it starts again in the next cycle at T (the end moment of the PDC) with the buyer's final inventories q_0. The idle time is equal to $T-t^*$. We have

$$Q=\sum_{j=1}^{m} q_j, \quad T=\frac{Q}{D} \quad \text{and} \quad t^*=\frac{Q}{P}\left(=\frac{T}{\lambda}\right). \tag{1}$$

Let us denote (with the notation i=0, for the vendor and i=1 for the buyer):

$I_i(t)$=the inventory position at t just before the possible replenishment;

$I_1(0)=q_0$=the buyer's initial inventory position.

Therefore, for a moment $t\in[0,T]$ we have the following inventory positions:

$$I_0(t)=P\min\{t,t^*\}-\sum\{q_j \mid t_j<t\},\ I_0(t^+)=P\min\{t,t^*\}-\sum\{q_j \mid t_j\le t\},$$

$$I_1(t)=I_1(0)-Dt+\sum\{q_j \mid t_j<t\}, \quad \text{with } t_j\le t \text{ for } I_1(t^+) \tag{2}$$

and the individual total stock holding quantities in the cycle

$$\bar{I}_0=\int_0^T I_0(t)\,dt \quad \text{and} \quad \bar{I}_1=\int_0^T I_1(t)\,dt.$$

Let us note that even each of $\bar{I}_0$ and $\bar{I}_1$ depends on the structure of the schedule $\tilde{q}$, the common total stock holding quantity $\bar{I}=\bar{I}_0+\bar{I}_1$ depends only on q_0. The main problem consists of the allocation of total stored quantity $\bar{I}$ in vendor's and buyer's stocks.

The above assumptions A1–A3 have an inspiration in practice and they are natural assumptions in joint buyer and vendor coordination models. Also, one of the following properties was postulated in the literature:

3a. The buyer receives all deliveries just to run out of the stock.
3b. The vendor's stock becomes empty just past each delivery.
3c. All shipments are identical, i.e. $q_1=\cdots=q_m$.
3d. The vendor's inventory is zero just past some initial replenishment and the successive (uniform in size) deliveries are sent *as soon as it is possible*.
3e. The vendor's inventory is zero just past some initial replenishment and the successive (uniform in size) deliveries are sent *as late as it is possible*.

According to some important (for the theory and the practice) classes of policies with respect to the above properties, we say that a policy is:

- a Goyal's policy if it satisfies postulates 3a and 3b (Goyal, 1976);
- an equal size delivery policy if 3c and 3a (Lu, 1995);
- a Hill's policy if 3a and 3e (Hill, 1999)—it implies that the initial deliveries increase in sizes with the rate λ;

- a CS-policy if 3c and 3d (Braglia and Zavanella, 2003; Zanoni and Grubbström, 2004) or 3e instead of 3d for the case $h_0 < h_1$;

Among the types of polices presented above, all, except the CS-policies, were formulated and investigated for the case $h_0 < h_1$. Although the optimal policy belongs to the class of Hill's policies, the other classes are applied in practice and still are investigated in the theory.

2.2. The class of feasible schedules in a PDC

According to "consignment stock", in a feasible PDC schedule $\tilde{q} = [(q_1,t_1),\ldots,(q_m,t_m)]$ on $[0,T]$ we set

$$I_0(0)=0 \quad \text{and} \quad I_1(0)=q_0=\frac{q^v}{P}D=I_1(T) \quad \text{and}$$

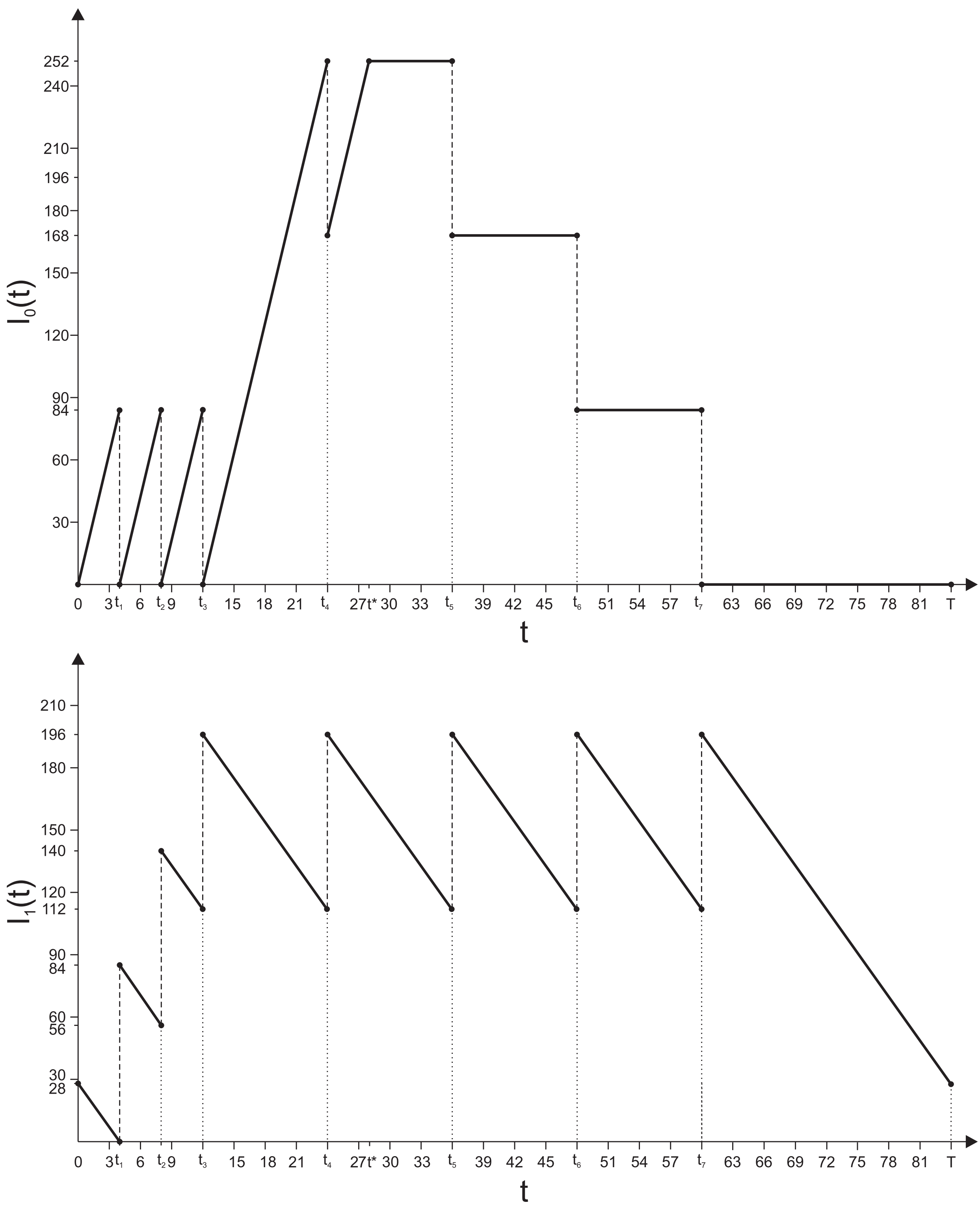

Fig. 1. The inventory positions for a schedule which satisfies 3c and 3d. The case of policy $k=3$, $n=4$ and $q^v=q^b=84$ with $I^b_{max}=196$, $t_{k+1}=t_k+q^b/D$.

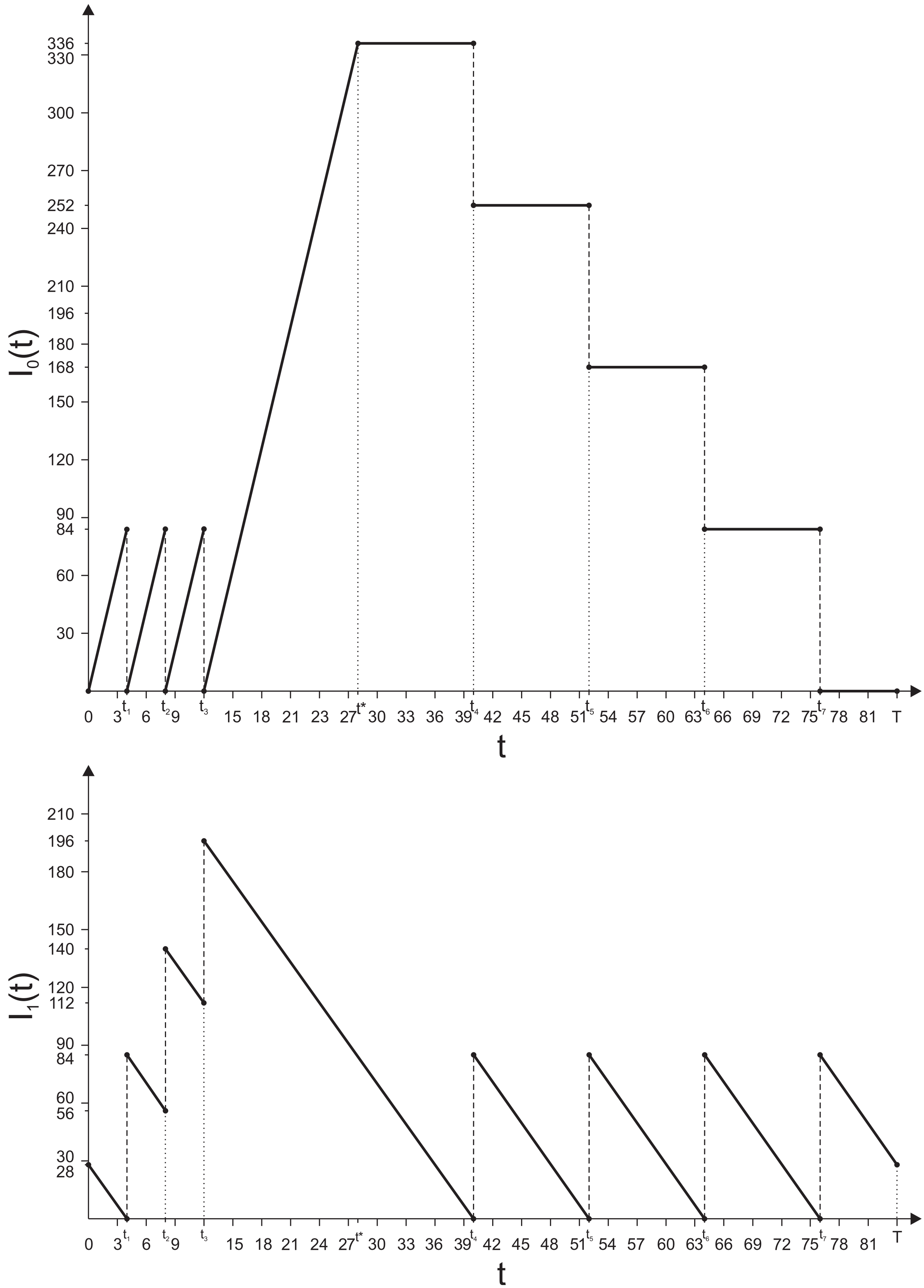

Fig. 2. The inventory positions for a schedule which satisfies 3c and 3e. The case of policy $k=3$, $n=4$ and $q^v=q^b=84$ with $I^b_{max}=196$, $t_{k+1}=t_k+I_1(t_k^+)/D$.

$$(q_j,t_j)=\begin{cases}(q^v,j\frac{q^v}{P}) & \text{for } j=1,\dots,k,\\ (q^b,t_{k+1}) & \text{for } j=k+1,\ n>0\\ (q^b,t_{k+1}+(j-k-1)\frac{q^b}{D}) & \text{for } j=k+2,\dots,k+n.\end{cases}\tag{3}$$

where $t_k+q^b/P\le t_{k+1}\le t_k+I_1(t_k^+)/D$ and $I_1(t_k^+)=q_0[1+k(\lambda-1)]$. Therefore each feasible PDC schedule can be determined by two pairs (q^v,k), (q^b,n) and, if $n>0$, a feasible time moment t_{k+1}. See Figs. 1 and 2 for feasible schedules with two different t_{k+1}.

In this paper we assume:

A3. *(more precise)* In a schedule of shipments in PDC, the buyer receives $m=k+n>0$ deliveries such that each of the following conditions hold:
1. The vendor's stock becomes empty just past each of $k\ge 0$ equal to q^v initial deliveries.
2. The last $n\ge 0$ equal to q^b deliveries are sent *as late as it is possible, i.e.* $t_{k+1}=t_k+I_1(t_k^+)/D$ *and* $I_1(t_j)=0$ *for* $j=k+1,\dots,k+n$.

Let us notice that our generalized CS-policies satisfy 3e which is more natural than 3d for the case $h_0<h_1$. For generalized CS policies, in opposite to Zanoni and Grubbström (2004), there is a possibility $q^v\ne q^b$. The delayed deliveries are sent as late as it is possible and the numbers of deliveries to be delayed are equal to n or $n-1$ (if $k=0$). Additionally, in the spirit of 3a buyer's stock is empty just before each of the last $n\ge 0$ deliveries.

With respect to the assumption A3, the production batch Q is partitioned as

$$Q=\sum_{j=1}^{k+n}q_j=kq^v+M,\quad \text{where } M=nq^b$$

which can be viewed as agent's decisions. Let us note that PDC schedules (given by Eq. (3)) can be determined by two pairs (Q,k) and (M,n) such that $Q\ge M$. It enables to determine the sizes of deliveries, particulary it ought to be $M=Q(n/k+n)$ in equal size delivery case.

It will be convenient for the presentation to set

$$k=0\quad\text{or } n=0\Longrightarrow q^v=q^b\ \left(\text{or } M=Q\frac{n}{k+n}\right).\tag{4}$$

We have the following maximal value of buyer's inventory level (the same as in the Zanoni and Grubbström (2004)):

$$I^b_{max}=\max\left\{q_0+kq^v-\frac{kq^v}{P}D,q^b\right\}.\tag{5}$$

With respect to the assumptions A1–A3, we are looking for a "good" schedule as a PDC. Each such feasible cycle is determined by two feasible pairs (q^v,k) and (q^b,n), with $k+n>0$ and Eq. (4). Additionally, it ought to be

$$P\left(\frac{q_0+kq^v}{D}-\frac{kq^v}{P}\right)\ge q^b\quad\text{and}\quad q_0\le q^b.$$

Therefore, the feasibility means: Eq. (4) and if $kn>0$, then

$$\frac{1}{\lambda}\le\frac{q^b}{q^v}\le 1+k(\lambda-1)=g_k\quad\left(\text{or }\frac{k+ng_k}{ng_k}\le\frac{Q}{M}\le\frac{n+k\lambda}{n}\right).\tag{6}$$

Formally, we can define the class of generalized CS-policies as

$$\Xi=\{((q^v,k),(q^b,n))\in(\mathcal{R}_+\times\mathcal{N})\times(\mathcal{R}_+\times\mathcal{N})\,|\ \text{such that (4) and (6)}\}$$

$$\text{or } \Pi=\{((Q,k),(M,n))\in(\mathcal{R}_+\times\mathcal{N})\times(\mathcal{R}_+\times\mathcal{N})\,|\text{such that (4) and (6)}\},$$

because each two pairs $((q^v,k),(q^b,n))\in\Xi$ (or $((Q,k),(M,n))\in\Pi$) determine the adequate schedule given by Eq. (3).

Table 1
The inventory positions $I_0(t)$ and $I_1(t)$ for the schedule with $(q^v,\ k)=(84,3)$ and $(q^b,\ n)=(42,4)$.

	$t_0\dots$	$t_1\dots$	$t_2\dots$	$t_3\dots$	$t^*\dots t_4$
t	$0\dots$	$4\|4^+\dots$	$8\|8^+\dots$	$12\|12^+\dots$	$20\dots 40$
I_0	$0\nearrow$	$84\|0\nearrow$	$84\|0\nearrow$	$84\|0\nearrow$	$168\to 168$
I_1	$28\searrow$	$0\|84\searrow$	$56\|140\searrow$	$112\|196\searrow$	$140\searrow 0$
	$t_4^+\dots$	$t_5\dots$	$t_6\dots$	$t_7\dots$	T
t	$40^+\dots$	$46\|46^+\dots$	$52\|52^+\dots$	$58\|58^+\dots$	60
I_0	$126\to$	$126\|84\to$	$84\|42\to$	$42\|0\to$	0
I_1	$42\searrow$	$0\|42\searrow$	$0\|42\searrow$	$0\|42\searrow$	28

Example 1. Let us consider production–distribution system such that $P=21$, $D=7$, and consider two policies

$$\xi=((q^v,k),(q^b,n))=((84,3),(42,4))\quad\text{and}\quad\overline{\xi}=((84,3),(84,4))$$

$$\pi=((Q,k),(M,n))=((420,3),(168,4)),\quad\overline{\pi}=((588,3),(336,4)).$$

We have $\lambda=3$ and

$$T=60,\ t^*=20\quad\text{and}\quad\overline{T}=84,\ \overline{t}^*=28$$

The varying of inventory positions in the schedules $q(\xi)$ and $q(\overline{\xi})$ (or $q(\pi)$ and $q(\overline{\pi})$) are given in Table 1 and Fig. 2, respectively.

Let us note that the assumption A3 and Eq. (6)—the feasibility conditions—hold. Therefore, both $\xi,\overline{\xi}\in\Xi$ (as well as $\pi,\overline{\pi}\in\Pi$).

From now on, Ξ_k^n (or Π_k^n) denotes the class of CS(k,n) policies with given $k\ge 0$ and $n\ge 0$, i.e.:

$$\Xi_k^n=\{(q^v,q^b)\,|\,((q^v,k),(q^b,n))\in\Xi\}\ \text{ or }\ \Pi_k^n=\{(Q,M)\,|\,((Q,k),(M,n))\in\Pi\}.\tag{7}$$

We can look for optimal generalized CS-policy for the integrated production–distribution problem equivalently in the set Ξ or Π. We do it in the set Ξ.

3. The cost of CS policies-analytical consideration

In this section we assume a central decision maker can minimize the sum of average total cost of the vendor plus the customer cost.

For a policy $\xi=((q^v,k),(q^b,n))\in\Xi$ (or $(q^v,q^b)\in\Xi_k^n$) the production butch is equal to $Q=kq^v+nq^b$. The average cost $C(\xi)$ of the schedule $\tilde{q}(\xi)=[(q_1,t_1),\dots,(q_m,t_m)]$ (given by Eq. (3) on $[0,T]$, $T=Q/D$) can be written as a sum of individual costs–the vendor's cost, V_k^n and buyer's cost B_k^n:

$$\begin{aligned}C_k^n(q^v,q^b)&=V_k^n(q^v,q^b)+B_k^n(q^v,q^b)\\&=\frac{1}{T}[A+kA_0+h_0\overline{I}_0(\xi)]+\frac{1}{T}[nA_1+h_1\overline{I}_1(\xi)],\end{aligned}$$

where

$$\overline{I}_i(\xi)=\int_0^T I_i(t)\,dt\quad\text{for } i=0,1.$$

With respect to the form of the functions $I_0(t)$ and $I_1(t)$, given by Eqs. (1)–(3) (see also Fig. 2 and Table 1), it is easy to calculate their integrals:

$$\overline{I}_0(\xi)=\frac{k(q^v)^2}{2P}+\frac{P(t^*-t_k)^2}{2}+(t^*-t_k)P(T-t^*)-\sum_{j=1}^{n}(T-t_{k+j})q^b\tag{8}$$

and

$$\overline{I}_1(\xi)=\int_{t_1}^{t_{k+1}}I_1(t)\,dt+\frac{n(q^b)^2}{2D}\quad\text{such that } t_{k+1}=T+\frac{q_0}{D}\ \text{if } n=0.\tag{9}$$

Let us first examine the formula (8). From Eqs. (1)–(3) we obtain

$$(t^{*}-t_k)=\frac{nq^b}{P} \quad \text{and} \quad \sum_{j=1}^{n}(T-t_{k+j})=\frac{1}{D}\sum_{j=1}^{n}(jq^b-q_0) = \frac{1}{D}\left[\frac{n(n+1)}{2}q^b-nq_0\right].$$

Therefore

$$\bar{I}_0(\xi)=\frac{k(q^v)^2-n^2(q^b)^2-2nkq^vq^b}{2P}+\frac{2knq^vq^b+n(n-1)(q^b)^2}{2D}+\frac{nq_0q^b}{D}.$$

For the vendor's cumulative inventory we have

$$\bar{I}_0(\xi)=\frac{1}{2P}[c_0(q^v)^2+b_0q^vq^b+a_0(q^b)^2], \quad \text{where}$$

$$c_0=k,\ b_0=\begin{cases}2n[k(\lambda-1)+1] & \text{for } k>0\\ 0 & \text{for } k=0\end{cases} \tag{10}$$

and

$$a_0=\begin{cases}(\lambda-1)n^2-\lambda n & \text{for } k>0,\\ (\lambda-1)n^2+(2-\lambda)n & \text{for } k=0.\end{cases}$$

According to the formula Eq. (9), from Eqs. (1)–(3) we obtain

$$\begin{aligned}\int_{t_1}^{t_{k+1}} I_1(t)\,dt &= \sum_{j=1}^{k-1}(t_{j+1}-t_j)\frac{I_1(t_j^+)+I_1(t_{j+1})}{2}+\frac{1}{2D}I_1(t_k^+)I_1(t_k^+)\\ &=\frac{q^v}{P}\sum_{j=1}^{k-1}\frac{2j(q^v-q_0)+q_0}{2}+\frac{1}{2D}[k(q^v-q_0)+q_0]^2\\ &=\frac{(q^v)^2}{2P\lambda}\langle k(k-1)(\lambda-1)+k-1+[k(\lambda-1)+1]^2\rangle\\ &=\frac{(q^v)^2}{2P}[k(\lambda-1)+1].\end{aligned}$$

Therefore, the buyer's cumulative inventory has the following quadratic formula:

$$\bar{I}_1(\xi)=\frac{1}{2P}[c_1(q^v)^2+b_1q^vq^b+a_1(q^b)^2], \quad \text{where}$$

$$c_1=k[k(\lambda-1)+1],\ b_1=0,\ a_1=\lambda n. \tag{11}$$

In the general case of the proposed CS(k,n) policies (with $k\geq 0$, $n\geq 0$, $m=k+n>0$) the total average cost take on the following form (by Eqs. (10) and (11))

$$\begin{aligned}C_k^n(q^v,q^b)&=C((q^v,k),(q^b,n)),\\ &=\frac{D}{Q}\left\langle \tilde{A}+\frac{h_0}{2P}[c_0(q^v)^2+b_0q^vq^b+a_0(q^b)^2]\right.\\ &\quad\left.+\frac{h_1}{2P}[c_1(q^v)^2+b_1q^vq^b+a_1(q^b)^2]\right\rangle,\end{aligned} \tag{12}$$

where (q^v,q^b) satisfies Eq. (6) and

$$\tilde{A}=A+kA_0+nA_1 \quad \text{and} \quad Q=kq^v+nq^b.$$

Definition 1. For a given pair (k,n), a policy $(\tilde{q}^v,\tilde{q}^b)$ is said to be optimal CS-(k,n) policy if it minimizes the cost given in Eq. (12) in the set Ξ_k^n.

Remark 1. To solve optimization problems with unique variable we use the Grubbström (1996) idea for the standard inventory lot size problems. If it is possible, rewrite the cost function $f(x)$ with x being the decision variable in the following way:

$$f(x)=ax+\frac{b}{x}+c=\frac{a}{x}\left(x-\sqrt{\frac{b}{a}}\right)^2+2\sqrt{ab}+c,$$

where a and b are positive coefficients. Therefore, the minimization solution must be $x^*=\sqrt{b/a}$, and $f(x^*)=2\sqrt{ab}+c$ will be the minimum value of $f(x)$.

Additionally, we can find a minimization solution in the set of natural numbers $\mathcal{N}$.

This simply algebraic procedure has been used in order to find optimal policies and the optimal cost value (without using the differential calculus) in many differ optimization problems. See Wu and Ouyang (2003) as an example.

4. Optimal CS(*k*,*n*)-policies

The cost function $C_k^n(q^v,q^b)$ given by Eq. (12) is to be minimized by a suitable choice of the two decision variables q^v and q^b. With respect to the form of the cost $C_k^n(q^v,q^b)$, it is useful to look at the cost as the function $\tilde{C}_k^n(q,\alpha)$ with the decision variables (q,α) such that

$$q^v=q>0, \quad q^b=\alpha q, \quad \text{whereby (6)}\ \frac{1}{\lambda}\leq\alpha\leq 1+k(\lambda-1)=g_k. \tag{13}$$

The function $\tilde{C}_k^n(q,\alpha)=C_k^n(q,\alpha q)$, by Eq. (12), has the form

$$\begin{aligned}\tilde{C}_k^n(q,\alpha)&=\frac{1}{q(k+n\alpha)}\left\langle \tilde{A}D+q^2\frac{1}{2\lambda}[c_0+b_0\alpha+a_0\alpha^2]h_0\right.\\ &\quad\left.+q^2\frac{1}{2\lambda}[c_1+b_1\alpha+a_1\alpha^2]h_1\right\rangle,\end{aligned}$$

as well as the form

$$\tilde{C}_k^n(q,\alpha)=\frac{1}{2\lambda}\left\langle\frac{2P\tilde{A}}{(k+n\alpha)q}+\frac{H_\alpha}{k+n\alpha}q\right\rangle, \quad \text{where}$$

$$H_\alpha=[c_0+b_0\alpha+a_0\alpha^2]h_0+[c_1+b_1\alpha+a_1\alpha^2]h_1. \tag{14}$$

This cost ought to be minimized by a suitable choice of the two decision variables q and α. For a given α it can be viewed as problem with q being the decision variable:

Theorem 1. *For each given $k\geq 0$, $n\geq 0$, $k+n>0$ and α satisfying Eq.* (13), *the policy with minimal cost* (*with respect to q*) *has the form* $(q^*(\alpha),\alpha)$, *where*:

$$q^*(\alpha)=\sqrt{\frac{2P\tilde{A}}{H_\alpha}} \quad \text{and} \quad C_\alpha^*(k,n)=\tilde{C}_k^n(q^*(\alpha),\alpha)=\frac{\sqrt{2P\tilde{A}H_\alpha}}{\lambda(k+n\alpha)}. \tag{15}$$

Proof. Consider the cost given in Eq. (14). To solve optimization problems with unique variable q we use the Grubbström's idea presented in Remark 1. For this reason the thesis of the theorem hold. □

Corollary 1. *For each pair (k,n) an optimal equal-size deliveries policy we obtain as a special case of Theorem1—to set $\alpha=1$ in Eq.* (15).

In the case k=0 or n=0 we have only one possibility for feasible α, namely, $\alpha=1$. In the other cases, to find a feasible α which minimizes the cost function $C_\alpha^*(k,n)$ we rewrite it as

$$C_\alpha^*(k,n)=\sqrt{\frac{2\tilde{A}D}{\lambda}}\sqrt{\varphi(\alpha)}, \quad \text{where } \varphi(\alpha)=\frac{p\alpha^2+r\alpha+s}{(k+n\alpha)^2} \text{ with}$$

$$\begin{aligned}p&=a_0h_0+a_1h_1=\lambda n(h_1-h_0)+(\lambda-1)h_0,\\ r&=b_0h_0=2n[k(\lambda-1)+1]h_0 \quad \text{and}\\ s&=c_0h_0+c_1h_1=kh_0+k[k(\lambda-1)+1]h_1.\end{aligned} \tag{16}$$

Lemma 1. *The function $\varphi(\alpha)$, given by Eq.* (16), *has not more than one stationary point. It is $\hat{\alpha}$*:

$$\hat{\alpha}=\frac{k^2(\lambda-1)(h_1-h_0)+kh_1}{k\lambda(h_1-h_0)-nh_0}, \quad \text{if} \quad \frac{h_1}{h_0}\neq\frac{k\lambda+n}{k\lambda} \text{ and } h_0\neq h_1.$$

We have

$$\hat{\alpha} > 0 \Leftrightarrow \frac{h_1}{h_0} > \frac{k\lambda+n}{k\lambda} \quad or \quad \frac{h_1}{h_0} < \frac{g_k-1}{g_k}$$

and the following three implications hold:

(a) *if* $h_1/h_0 > (k\lambda+n)/k\lambda$ *then* $\hat{\alpha}$ *is the local minimum of* $\varphi(\alpha)$;
(b) *if* $h_1/h_0 < (g_k-1)/g_k$ *then* $\hat{\alpha}$ *is the local maximum of* $\varphi(\alpha)$;
(c) *if* $(g_k-1)/g_k \le h_1/h_0 \le (k\lambda+n)/k\lambda$ *then* $\varphi(\alpha)$ *decreases for* $\alpha > 0$.

Proof. It is easy to find

$$\varphi'(\alpha) = \frac{(2p\alpha+r)(k+n\alpha)^2 - 2(k+n\alpha)(p\alpha^2+r\alpha+s)}{(k+n\alpha)^4} = \frac{(2kp-nr)\alpha + kr - 2\,ns}{(k+n\alpha)^3}$$

which can be write as

$$\varphi'(\alpha) = \frac{I\alpha + J}{(k+n\alpha)^3}, \text{ where}$$

$$I = 2kp - nr = 2k\langle[(\lambda-1)n^2 - \lambda n]h_0 + \lambda n h_1\rangle - 2n^2[k(\lambda-1)+1]h_0 = 2k\lambda n(h_1-h_0) - 2n^2 h_0$$

and

$$J = kr - 2ns = 2kn[k(\lambda-1)+1]h_0 - 2n\langle kh_0 + k[k(\lambda-1)+1]h_1\rangle = -2nkg_k(h_1-h_0) - 2nkh_0.$$

Therefore

$$\varphi'(\alpha) = 0 \;\Rightarrow\; \alpha = \frac{-J}{I} = \hat{\alpha}$$

and it is the first part of the Lemma.

Our next claim is that

$$I > 0 \Leftrightarrow 2k\lambda n(h_1-h_0) > 2n^2 h_0 \Leftrightarrow \frac{h_1}{h_0} > \frac{k\lambda+n}{k\lambda}$$

with

$$I = 0 \Leftrightarrow \frac{h_1}{h_0} = \frac{k\lambda+n}{k\lambda}$$

and

$$J > 0 \Leftrightarrow 2nkg_k(h_0-h_1) > 2nkh_0 \Leftrightarrow \frac{h_1}{h_0} < \frac{g_k-1}{g_k},$$

with

$$J = 0 \Leftrightarrow \frac{h_1}{h_0} = \frac{g_k-1}{g_k}.$$

A trivial verification shows that the thesis (a)–(c) of the Lemma hold. □

Theorem 2. *Assume* $k > 0$ *and* $n > 0$. *Consider* $q^*(\alpha)$ *given by Eq.* (15) *and* $\hat{\alpha}$ *determined in Lemma1. Define* $(\tilde{q}^v, \tilde{q}^b) \in \Xi_k^n$ *such that*

$$\tilde{q}^v = q^*(\alpha^*) \quad and \quad \tilde{q}^b = \alpha^* q^*(\alpha^*),$$

where α^* *is defined in the following way*:

$$\alpha^* = \begin{cases} \frac{1}{\lambda} & \text{if } \frac{h_1}{h_0} \le \frac{k\lambda+n}{k\lambda} \text{ and } \varphi(\frac{1}{\lambda}) \le \varphi(g_k), \\ \frac{1}{\lambda} & \text{if } \frac{h_1}{h_0} > \frac{k\lambda+n}{k\lambda} \text{ and } \hat{\alpha} < \frac{1}{\lambda}, \\ \hat{\alpha} & \text{if } \frac{h_1}{h_0} > \frac{k\lambda+n}{k\lambda} \text{ and } \frac{1}{\lambda} \le \hat{\alpha} \le g_k, \\ g_k & \text{otherwise.} \end{cases}$$

The policy $(\tilde{q}^v, \tilde{q}^b) \in \Xi_k^n$ *is an optimal CS*(k,n)*-policy.*

Proof. With respect to Eqs. (13)–(16) it is enough to find feasible α^* which minimizes $\varphi(\alpha)$ in the interval $[1/\lambda, g_k)]$. The three possibilities on α^* given in the thesis of the theorem are consequences of Lemma 1. Namely, in the case $h_1/h_0 > (k\lambda+n)/k\lambda$ we have

$$\alpha^* = \begin{cases} \frac{1}{\lambda} & \text{if } \hat{\alpha} \le \frac{1}{\lambda}, \\ \hat{\alpha} & \text{if } \frac{1}{\lambda} \le \hat{\alpha} \le g_k \\ g_k & \text{if } \hat{\alpha} \ge g_k. \end{cases}$$

In the case $h_1/h_0 < (g_k-1)/g_k$ we have

$$\alpha^* = \begin{cases} g_k & \text{if } \hat{\alpha} \le \frac{1}{\lambda}, \\ \frac{1}{\lambda} & \text{if } \frac{1}{\lambda} \le \hat{\alpha} \le g_k,\ \varphi(\frac{1}{\lambda}) \le \varphi(g_k) \\ g_k & \text{if } \frac{1}{\lambda} \le \hat{\alpha} \le g_k,\ \varphi(\frac{1}{\lambda}) \ge \varphi(g_k) \\ \frac{1}{\lambda} & \text{if } \hat{\alpha} \ge g_k. \end{cases}$$

In the case $(g_k-1)/g_k \le h_1/h_0 \le (k\lambda+n)/k\lambda$ we have $\alpha^* = g_k$.

This completes the proof, the detailed verification of the formulas for α^* in the thesis of the theorem is obvious, easy. □

Table 2
Optimal CS-(k,n) policies for $m = k+n$ equal to $2, 3, \ldots, 7$.

(k,n)	$\tilde{q}^v$	$\tilde{q}^b$	V_k^n	B_k^n	$C^*(k,n)$	$q^*(1)$	V_k^n	B_k^n
(1,1)	113,2	362,2	1083,33	809,86	1893,19	223,6	1397,5	614,9
(1,2)	70,6	225,9	1210,21	608,06	1818,27	164,2	1417,0	511,9
(2,1)	68,6	370,7	**1066,56**	803,7	1870,26	161,1	1327,3	638,9
(1,3)	52,2	167,0	1281,82	525,98	**1807,8**	131,3	1432,9	471,1
(2,2)	42,9	231,5	1214,96	607,64	1822,6	129,8	1394,1	582,3
(3,1)	50,0	379,8	1075,69	811,99	1887,68	126,9	1276,9	693,5
(1,4)	41,8	133,8	1327,89	491,19	1819,08	110,3	1446,2	457,1
(2,3)	31,7	170,0	1296,7	525,31	1822,01	109,5	1432,4	485,9
(3,2)	31,2	237,1	1235,87	613,59	1849,46	107,8	1363,7	584,6
(4,1)	53,4	335,3	1195,44	716,73	1912,17	105,4	1246,5	745,8
(1,5)	35,2	112,5	1360,31	479,05	1839,96	95,7	1457,9	456,9
(2,4)	25,3	136,9	1348,6	489,48	1838,08	95,2	1458,2	467,6
(3,3)	23,0	175,0	1323,23	528,72	1851,95	94,1	1417,6	529,9
(4,2)	24,7	242,6	1261,42	621,42	1882,84	92,6	1339,7	639,9
(5,1)	48,5	323,7	1240,43	701,59	1942,02	90,7	1229,4	792,2
(1,6)	30,5	97,6	1386,53	479,16	1865,69	85,0	1468,7	464,6
(2,5)	21,3	115,0	1385,33	**476,29**	1861,62	84,6	1477,5	464,1
(3,4)	18,4	140,0	1378,34	490,91	1869,25	83,9	1454,5	503,6
(4,3)	18,2	178,9	1352,92	533,4	1886,32	82,8	1402,1	580,6
(5,2)	24,0	239,8	1310,38	607,95	1918,33	81,5	1323,2	691,8
(6,1)	44,2	317,4	1276,04	698,34	1974,38	79,9	1221,1	893,5

4.1. Numerical example

We use Goyal's numerical example to present our solution procedure for CS(k,n) policies. The total cost found by the method in this study can be compared with the costs found in many papers, where the same numerical example was tested.

The data for the example are:

$A = 400$ $A_0 = A_1 = 25$ $h_0 = 4$ $h_1 = 5$ $D = 1000$ $P = 3200$.

For this example, by Theorem 2, we determine the individual and system costs of optimal CS-(k,n) policies. Also, optimal equal-size policies, given as $q^*(1)$ in Theorem 1 and Corollary 1, are presented in Table 2.

It is easy to countable the total cost of equal-size policies and to check that in each case the optimal CS(k,n) policy reduces the system cost about 5%. Both agents participate in this benefice.

5. Individual strategies in non-cooperative case

In this section we look at generalized CS-policies as pairs of individual strategies in non-cooperative 2-players game. Namely $((q^v,k),(q^b,n)) \in \Xi$ as a pair of individual vendor and buyer strategies in a game, say G, restricted to Ξ. Analogously $((Q,k),(M,n)) \in \Pi$ as a pair strategies in a game, say Γ, restricted to Π. From now on we make the assumption that the numbers of deliveries $k>0$ and $n>0$ are fixed before the game, for example, by negotiations. It leads to the game $G_k^n=(\Xi_k^n, V_k^n, B_k^n)$, a sub-game of the game G or the game Γ_k^n, a sub-game of Γ. Additionally, we assume that the agents decide about own butches trough a non-cooperative constrained chose.

Remark 2. In this section will be considered carefully the sub-games Γ_k^n. The vendor decides about butch quantity Q produced in the cycle (so on about the length of the cycle T). The buyer decides about the part $M<Q$ which will be delivered on the buyer's mode (delayed with respect to vendor's point of view). Each pair of butch quantities (Q,M), with $0<M<Q$, determines the sizes of successive deliveries $(q^v(Q,M), q^b(Q,M))$:

$$q^v(Q,M)=\frac{Q-M}{k} \quad \text{and} \quad q^b(Q,M)=\frac{M}{n}, \tag{17}$$

if only the feasible condition in Eq. (6) be satisfied.

5.1. Analytical consideration

For given $k>0$ and $n>0$, let us define analogously as Bylka (2009) the constrained game $\Gamma_k^n=(\Pi_k^n, \hat{V}_k^n, \hat{B}_k^n)$, where Π_k^n is the set of feasible pairs of strategies given by Eq. (7). It is easy to check, by Eq. (6) that Π_k^n is a convex cone in $\mathcal{R}_+ \times \mathcal{R}_+$. Additionally, for each "position" $(Q,M) \in \Pi_k^n$ the sets of possible responses of the vendor and the buyer on partner's strategies are closed intervals.

For $(Q,M) \in \Pi_k^n$ we define

$$\hat{V}_k^n(Q,M)=V_k^n(q^v(Q,M),q^b(Q,M)) \quad \text{and}$$
$$\hat{B}_k^n(Q,M)=B_k^n(q^v(Q,M),q^b(Q,M))$$

as the vendor and the buyer costs, respectively. In addition, all model parameters are common knowledge.

By Eqs. (17) and (12) we obtain

$$\hat{V}_k^n(Q,M)=\frac{1}{Q}\left\langle \tilde{A}_0 D+\frac{h_0}{2\lambda}\left[c_0\frac{(Q-M)^2}{k^2}+b_0\frac{(Q-M)M}{kn}+a_0\frac{M^2}{n^2}\right]\right\rangle,$$

where $\tilde{A}_0=A+kA_0$,

$$\hat{B}_k^n(Q,M)=\frac{1}{Q}\left\langle \tilde{A}_1 D+\frac{h_1}{2\lambda}\left[c_1\frac{(Q-M)^2}{k^2}+a_1\frac{M^2}{n^2}\right]\right.$$

where $\tilde{A}_1=nA_1$, (18)

Definition 2. A pair of strategies $(Q^*,M^*) \in \Pi_k^n$ is called an equilibrium in Γ_k^n iff we have

$\hat{V}_k^n(Q^*,M^*) \le \hat{V}_k^n(Q,M^*)$ for every $(Q,M^*) \in \Pi_k^n$

and

$\hat{B}_k^n(Q^*,M^*) \le \hat{B}_k^n(Q^*,M)$ for every $(Q^*,M) \in \Pi_k^n$.

- Is there a CS(k,n) policy which is an equilibrium in the game Γ_k^n ?

A positive answer which concerns the Hill's policies in analogous game were given by Bylka (2009). For the CS(k,n) policies we have the following positive results:

Theorem 3. *For each pair of positive natural numbers (k,n) the pair (Q^*,M^*) such that*

$$Q^*=\sqrt{\tilde{\mu}\frac{2\lambda\tilde{A}_0 D}{h_0}} \quad \textit{and} \quad M^*=\mu^* Q^*, \textit{ where}$$

$$\mu^*=\frac{ng_k}{k\lambda+ng_k} \quad \textit{and} \quad \tilde{\mu}=\frac{k}{1+g_k\mu^*}$$

is the one and only one pure equilibrium in the game Γ_k^n.

Proof. For every $Q>0$, for the buyer's cost as a function of M we have

$$\frac{\partial \hat{B}_k^n(Q,M)}{\partial M}=\frac{h_1}{2Q\lambda}\left[-2c_1\frac{Q-M}{k^2}+2a_1\frac{M}{n^2}\right]$$
$$=\frac{h_1}{Q\lambda k^2 n^2}[(a_1k^2+c_1n^2)M-c_1n^2Q].$$

It is a linear increasing function of M and

$$\frac{\partial \hat{B}_k^n(Q,M)}{\partial M}\ge 0 \Leftrightarrow M \ge \frac{c_1n^2}{a_1k^2+c_1n^2}Q=\mu^* Q.$$

Therefore, $\hat{B}_k^n(Q,M)$ as a function of M take minimum for $M=\mu^* Q$. The conditions on feasibility of strategies, by Eq. (6), postulate

$$\frac{k+ng_k}{ng_k} \le \frac{Q}{\mu^* Q} \le \frac{k\lambda+n}{n}.$$

It requires to be

$$\frac{k+ng_k}{ng_k} \le \frac{k\lambda+ng_k}{ng_k} \le \frac{k\lambda+n}{n} = \frac{k\lambda g_k+ng_k}{ng_k}.$$

Fortunately, the above inequalities are true in any case. Therefore, for every Q the strategy

$\hat{M}(Q)=\mu^* Q$ is the buyer's best response (19)

on vendor's strategy Q.

For the vendor's cost as a function of Q we have

$$\frac{\partial \hat{V}_k^n(Q,M)}{\partial Q}=\frac{1}{Q^2}\left\langle\frac{h_0}{2\lambda}\left[\frac{2c_0(Q-M)}{k^2}+\frac{b_0M}{kn}\right]Q-\tilde{A}_0D\right.$$
$$\left.-\frac{h_0}{2\lambda}\left[\frac{c_0(Q-M)^2}{k^2}+\frac{b_0M(Q-M)}{kn}+\frac{a_0M^2}{n^2}\right]\right\rangle$$
$$=\frac{h_0}{2\lambda Q^2}\left[\frac{c_0(Q^2-M^2)}{k^2}+\frac{b_0M^2}{kn}-\frac{a_0M^2}{n^2}-\frac{2\lambda\tilde{A}_0D}{h_0}\right]$$
$$=\frac{h_0}{2\lambda Q^2}\left[\frac{Q^2}{k}+\rho M^2-\frac{2\lambda\tilde{A}_0D}{h_0}\right],$$

where

$$\rho=\frac{-c_0n^2-a_0k^2+b_0kn}{k^2n^2}=\frac{n+k\lambda+(\lambda-1)nk}{kn}=\frac{k\lambda+ng_k}{kn}.$$

We have

$$\frac{\partial \hat{V}_k^n(Q,M)}{\partial Q}\ge 0 \Leftrightarrow \frac{Q^2}{k}+\rho M^2-\frac{2\lambda\tilde{A}_0D}{h_0}\ge 0.$$

With respect to the form of the above quadratic function of Q, the function $\hat{V}_k^n(Q,M)$ as a function of Q take minimum for

$$\overline{Q}(M)=\begin{cases} 0 & \text{if } \rho M^2 \ge \dfrac{2\lambda\tilde{A}_0D}{h_0}, \\ \sqrt{k\left(\dfrac{2\lambda\tilde{A}_0D}{h_0}-\rho M^2\right)} & \text{otherwise,} \end{cases} \tag{20}$$

The conditions on feasibility of strategies, by Eq. (6), require

$$\beta_0=\frac{k+ng_k}{ng_k} \le \frac{Q(M)}{M} \le \frac{k\lambda+n}{n}=\beta^0.$$

Therefore, by Eq. (20), the strategy

$$\hat{Q}(M) = \begin{cases} \beta_0 M & \text{if } \overline{Q}(M) < \beta_0 M, \\ \beta^0 M & \text{if } \overline{Q}(M) > \beta^0 M, \\ \overline{Q}(M) & \text{otherwise} \end{cases} \tag{21}$$

is the vendor's best response on buyer's strategy M.

For a Nash equilibrium (Q,M) it ought to be

$$Q = \hat{Q}(M) \quad \text{and} \quad M = \hat{M}(Q) = \mu^* Q.$$

It implies $Q = \hat{Q}(\mu^* Q)$ and so on

$$Q = \beta_0 \mu^* Q \quad \text{or} \quad Q = \overline{Q}(\mu^* Q) \quad \text{or} \quad Q = \beta^0 \mu^* Q$$

From the above equalities only the second one can be true because

$$\beta_0 \mu^* = \frac{k + n g_k}{k\lambda + n g_k} < 1 \quad \text{and} \quad \beta^0 \mu^* = \frac{k\lambda g_k + n g_k}{k\lambda + n g_k} > 1.$$

Our next goal is to determine the solution of the equation

$$Q = \sqrt{k\left(\frac{2\lambda \tilde{A}_0 D}{h_0} - \rho(\mu^* Q)^2\right)}.$$

As the unique solution we obtain

$$Q^* = \sqrt{\frac{k}{1 + k\rho(\mu^*)^2} \frac{2\lambda \tilde{A}_0 D}{h_0}} = \sqrt{\frac{k}{1 + g_k \mu^*} \frac{2\lambda \tilde{A}_0 D}{h_0}} = \sqrt{\hat{\mu} \frac{2\lambda \tilde{A}_0 D}{h_0}}.$$

It follows immediately that the pair $(Q^*, \mu^* Q^*)$ forms the unique equilibrium, and the proof is complete. □

5.2. Numerical example (continued)

For the example given in Section 4.1, we can find CS-(k,n) polices—the equilibria in the games Γ_k^n.

It is obvious that if (Q^*,M^*) is an equilibrium in the game Γ_k^n then the pair of strategies $q^v(Q^*,M^*), q^b(Q^*,M^*)$ need not to be an equilibrium pair in the game G_k^n, and vice versa.

In the same way as in the proof of Theorem 3 (Eqs. (19)–(21)) we can find explicit formulas for the best responds in the game G_k^n. Particulary, it can be checked that the pair $(q^{*v}, q^{*b}) = (164,6, 119,4)$ is an equilibrium in the game G_2^3 but the pair $(Q,M) = (2q^{*v} + 3q^{*b}, 3q^{*b}) = (687,5, 358,3)$ is not an equilibrium in the game Γ_2^3 (which has unique equilibrium given in Tables 3 and 4). Two examples of equilibrium strategies in G_k^n and Γ_k^n (for $(k,n) = (2,3)$ and $(k,n) = (3,3)$) are presented in Table 3 more precisely.

The equilibrium CS(k,n) strategies in the games Γ_k^n and their costs (additionally with the costs of the equilibrium Hill's policies for the analogous game considered by Bylka (2009)) are presented in Table 4.

6. Final remarks and conclusions

In this paper, we have studied two cases when decisions are centralize coordinated, and when the vendor and buyer are independent firms, each with the goal of minimizing their own total cost. Generalized consignment stock policy (pair of strategies (q^v,k) and (q^b,n)) admits two sizes of shipments q^v and q^b with regard to the vendor's and buyer's preferences. It is worth to consider relations q^v and q^b with respect to the vendor's EPQ and the buyer's EOQ. The arrival of the last of the early k deliveries determines the maximum stock level reached at the buyer. This quantity is very important in the context of consignment inventory. The problem with buyer's maximum stock level as an additional decision variable is not considered.

The CS(k,n)-policies can also be viewed from competition perspective—as a pair of non-cooperative vendor–buyer strategies for production–distribution system. Such policies $(q^v,q^b) \in \Xi_k^n$ can be used as strategies in (k,n)-sub-games of a non-cooperative constrained game G. On the other hand, each CS(k,n)-policies can also be viewed (because identical schedules and costs) as a pair $(Q,M) \in \Pi_k^n$. The presented transformation of policies and spaces is natural in the cooperation case and leads to the same optimal solutions. In the non-cooperative case we obtain another game Γ. The sub-games Γ_k^n with policies $(Q,M) \in \Pi_k^n$ as pair of strategies

Table 4
Equilibrium strategies in the games G_k^n and Γ_k^n for $k+n \le 7$.

In Γ_k^n	For CS(k,n) strategies					For Hill's	Strategies
(k,n)	Q^*	M^*	$\hat{V}_k^n$	$\hat{B}_k^n$	$C^*(k,n)$	B_k^n	$C^*(k,n)$
(1,1)	511,4	255,4	1342,45	688,14	2030,59	688,14	2030,59
(1,2)	465,8	310,6	1436,39	495,54	1931,93	495,54	1931,93
(2,1)	644,1	294,7	1213,1	775,7	1988,8	725,47	1912,0
(1,3)	447,2	335,4	1481,4	447,21	1928,61	447,21	1928,61
(2,2)	572,7	359,6	1346,8	536,8	1883,6	524,1	1833,0
(3,1)	723,3	319,6	1180,26	833,56	2013,82	744,68	1910,74
(1,4)	437,0	349,6	1507,82	447,33	1955,15	447,33	1955,15
(2,3)	543,7	389,7	1411,65	462,73	**1874,38**	454,51	**1827,48**
(3,2)	634,8	389,1	1334,44	565,1	1899,54	539,62	1839,75
(4,1)	780,7	338,6	**1175,02**	878,41	2053,43	762,32	1942,29
(1,5)	430,6	358,9	1525,2	469,7	1994,9	469,7	1994,9
(2,4)	528,0	407,3	1450,05	**443,96**	1894,02	**433,58**	1846,11
(3,3)	599,3	421,7	1409,46	476,59	1886,05	462,73	1834,99
(4,2)	679,6	411,1	1342,96	587,47	1930,43	551,68	1871,88
(5,1)	826,9	354,4	1181,25	916,17	2097,42	779,63	1983,54
(1,6)	426,2	365,3	1537,5	504,14	2041,65	504,14	2041,65
(2,5)	518,1	418,8	1475,48	450,68	1926,16	436,21	1875,6
(3,4)	580,1	440,9	1454,04	447,93	1901,97	434,65	1852,71
(4,3)	639,4	445,4	1424,79	488,51	1913,3	470,0	1866,29
(5,2)	715,7	429,4	1359,79	606,62	1966,41	562,59	1911,06
(6,1)	866,2	368,3	1193,19	949,53	2142,71	796,61	2027,2

Table 3
Equilibrium strategies in the games G_k^n and Γ_k^n for $(k, n) = (2,3)$ and $(3,3)$.

In G_k^n	Equilibrium strategies						
(k,n)	q^{*v}	q^{*b}	Q	M	V_k^n	B_k^n	$C^*(k,n)$
(2,3)	164,605	119,444	687,542	358,332	1415,123	597,220	2012,343
(3,3)	132,391	130,588	658,349	261,176	1406,076	652,912	2058,988
In Γ_k^n	Equilibrium strategies						
(k,n)	q^v	q^b	Q^*	M^*	V_k^n	B_k^n	$C^*(k,n)$
(2,3)	76,988	129,917	543,727	389,752	1411,646	462,730	1874,376
(3,3)	59,191	140,576	599,300	421,729	1409,464	476,587	1886,051

(analogously as Hill's policies in Bylka (2009)) were Investigated, and explicit formulas for Nash equilibrium strategies in sub-games were presented.

The main goal of this paper was to show that satisfactory coordination of inventories can be achieved using a competitive approach such as the framework of non-cooperative games. The presented games are different in the decision rules (i.e. in the equivalent positions the agents have different sets of potential moves). The equilibrium strategies in such games, as well as the total costs and agents' participation in such costs, are different. In numerical simulations we observe that the total costs of equilibrium strategies in the games Γ_k^n are close to the total costs of optimal policies in integrated system. Let us note that the considered game Γ_k^n seems to be more restricted than the game G_k^n with q^v and q^b as strategies. Theoretical justification of them is beyond the scope of the paper.

The problem of existence of equilibrium strategies in the games G_k^n remains open.

One question still unanswered is whether there exists an equilibrium in the main constrained game Γ or the game G.

References

Banerjee, A., 1986. A joint economic-lot-size model for purchaser and vendor. Decision Sciences 17, 292–311.

Braglia, M., Zavanella, L., 2003. Modelling an industrial strategy for inventory management in supply chains: the ''Consignment Stock'' case. International Journal of Production Research 41, 3793–3808.

Bylka, S., 2009. Non-cooperative strategies for production and shipments lot sizing in the vendor–buyer system. International Journal of Production Economics 118, 243–252.

Chen, J.M., Lin, I.C., Cheng, H.L., 2010. Channel coordination under consignment and vendor-managed inventory in a distribution system. Transportation Research Part E 46, 831–843.

Goyal, S.K., 1976. An integrated inventory model for a single supplier single customer system. International Journal of Production Research 15, 107–111.

Grubbström, R.W., 1996. Material requirements planning and manufacturing resource planning. In: Werner, M. (Ed.), International Encyclopedia of Business and ManagementRoutledge, London, pp. 3400–3420.

Gümüs, M., Jewkes, E.M., Bookbinder, J.H., 2008. Impact of consignment inventory and vendor-managed inventory for a two-party supply chain. International Journal of Production Economics 113, 502–517.

Hill, R.M., 1999. The optimal production and shipment policy for the single-vendor single-buyer integrated production inventory problem. International Journal of Production Research 37, 2463–2475.

Lu, L., 1995. A one-vendor multi-buyer integrated inventory model. European Journal of Operational Research 81, 312–324.

Ru, J., Wang, Y., 2010. Consignment contracting: who should control inventory in the supply chain. European Journal of Operational Research 201, 760–769.

Sarmah, S.P., Acharya, D., Goyal, S.K., 2006. Buyer–vendor coordinated models in supply chain management. European Journal of Operational Research 175, 1–15.

Wu, K.-S., Ouyang, L.-Y., 2003. An integrated single-vendor single-buyer inventory system with shortage derived algebraically. Production Planning and Control 14 (6), 555–561.

Zanoni, S., Grubbström, R.W., 2004. A note on industrial strategy for stock management in supply chains: modelling and performance evaluation. International Journal of Production Research 42, 4421–4426.

Zavanella, Z., Zanoni, S., 2009. A one-vendor multi-buyer integrated production-inventory model: The 'Consignment Stock' case. International Journal of Production Economics 118, 225–232.